AN

INTRODUCTION
TO
ALGEBRA;

WITH

NOTES AND OBSERVATIONS:

DESIGNED FOR THE USE OF SCHOOLS
AND PLACES OF PUBLIC EDUCATION.

BY JOHN BONNYCASTLE,
OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

THE SECOND EDITION.

—— Ingenuas didiciſſe fideliter artes
Emollit mores, nec ſinit eſſe feros. OVID.

LONDON:
PRINTED FOR J. JOHNSON, ST. PAUL'S CHURCH YARD.
MDCCLXXXVIII.

BRITISH MUSEUM
15 MAY 68

TO THE REVEREND

J. PRIESTLEY, LL.D. F.R.S. &c.

SIR,

THE publication of the following treatiſe is owing to your kind encouragement and approbation; and I am happy to embrace this opportunity of teſtifying the high ſenſe I entertain of your condeſcending politeneſs and attention. Whilſt you are enlarging the bounds of ſcience by your numerous and important diſcoveries, you are equally deſirous of promoting every other laudable purſuit and uſeful undertaking. And to this amiable diſpoſition the world is no leſs indebted than to your diſtinguiſhed and eminent abilities: the one commands our eſteem and regard, and the other our admiration. Permit me, therefore, Sir, as a ſincere tribute to your merit, to inſcribe to you this compendium, and to aſſure you that

I am,

With the higheſt reſpect,

Your moſt obedient and

obliged humble ſervant,

JOHN BONNYCASTLE.

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PREFACE.

THE powers of the mind, like thoſe of the body, are increaſed by frequent exertion; application and induſtry ſupply the place of genius and invention; and even the creative faculty itſelf may be ſtrengthened and improved by uſe and perſeverance. Uncultivated nature is uniformly rude and imbecile; and it is by imitation alone that we at firſt acquire knowledge, and the means of extending its bounds. A juſt and perfect acquaintance with the ſimple elements of ſcience is a neceſſary ſtep towards our future progreſs and advancement; and this, aſſiſted by laborious inveſtigation and habitual inquiry, will conſtantly lead to eminence and perfection.

Books of rudiments, therefore, conciſely written, well digeſted, and methodically arranged, are treaſures of ineſtimable value; and too many attempts cannot be made to render them perfect and complete. When the firſt principles of any art or ſcience are firmly fixed and rooted in the mind, their application ſoon becomes eaſy, pleaſant, and obvious; the underſtanding is delighted and enlarged; we conceive clearly, reaſon diſtinctly, and form juſt and ſatisfactory concluſions. But, on the contrary, when the mind, inſtead of repoſing on the ſtability of truth and received principles, is wandering in doubt and uncertainty, our ideas will neceſſarily be confuſed and obſcure; and every ſtep we take muſt be attended with freſh difficulties and endleſs perplexity.

That the grounds, or fundamental parts, of every ſcience, are dull and unentertaining, is a complaint univerſally made, and a truth not to be denied; but, then, what is obtained with difficulty is uſually remembered with eaſe; and what is purchaſed with pain is poſſeſſed with pleaſure. The ſeeds of knowledge are ſown in every ſoil, but it is by proper culture alone that they are cheriſhed and brought to maturity. A few years of early and aſſiduous application never fails of procuring us the reward of our induſtry; and who is there, that knows the pleaſures and advantages which the ſciences afford, that would think his time miſſpent, or his labours uſeleſs? Riches and honours are the gifts of fortune, caſually beſtowed, or hereditarily received, and are frequently abuſed by their poſſeſſors; but the ſuperiority of wiſdom and knowledge is a pre-eminence of merit, which originates with the man, and is the nobleſt of all diſtinctions.

Nature, bountiful and wiſe in all things, has provided us with an infinite variety of ſcenes, both for our inſtruction and entertainment; and, like a kind and indulgent parent, admits all her children to an equal participation of her bleſſings. But as the modes, ſituations, and circumſtances of life are various, ſo accident, habit, and education, have each their predominating influence, and give to every mind its particular bias. Where examples of excellence are wanting, the attempts to attain it are but ſew; but eminence excites attention, and produces imitation. To raiſe the curioſity, and to awaken the liſtleſs and dormant powers of younger minds, we have only to point out to them a valuable acquiſition, and the means of obtaining it. The active principles are immediately put into motion, and the certainty of the conqueſt is enſured from a determination to conquer.

But, of all the ſciences which ſerve to call forth this ſpirit of enterpriſe and inquiry, there are none more eminently uſeful than the Mathematics. By an early attachment to theſe elegant and ſublime ſtudies, we acquire a habit of reaſoning, and an elevation of thought, which fixes the mind, and prepares it for every other purſuit. From a few ſimple axioms, and evident principles, we proceed gradually to the moſt general propoſitions, and remote analogies: deducing one truth from another, in a chain of argument well connected and logically purſued; which brings us at laſt, in the moſt ſatisfactory manner, to the concluſion, and ſerves as a general direction in all our inquiries after truth.

And it is not only in this reſpect that mathematical learning is ſo highly valuable; it is, likewiſe, equally eſtimable for its practical utility. Almoſt all the works of art, and devices of man, have a dependence upon its principles, and are indebted to it for their origin and perfection. The cultivation of theſe admirable ſciences is, therefore, a thing of the utmoſt importance, and ought to be conſidered as a principal part of every liberal and well regulated plan of education. They are the guide of our youth, the perfection of our reaſon, and the foundation, or baſis, of every great and noble undertaking.

From theſe conſiderations, I have been induced to undertake an introductory courſe of mathematical ſcience; and, from the kind encouragement which I have hitherto received, am not without hopes of a continuance of the ſame candour and approbation. Conſiderable practice as a teacher, and a long attention to the difficulties and obſtructions which retard the progreſs of learners in general, have enabled me to accommodate myſelf the more eaſily to their capacities and underſtandings. And as an earneſt deſire of promoting and diffuſing uſeful knowledge is the chief motive for this undertaking, ſo no pains or attention ſhall be wanting to make it as complete and perfect as poſſible.

The ſubject of the preſent performance is ALGEBRA; which is one of the moſt important and uſeful branches of thoſe ſciences, and may be juſtly conſidered as the key to all the reſt. Geometry delights us by the ſimplicity of its principles, and the elegance of its demonſtrations. Arithmetic is confined in its object, and partial in its application: but Algebra, or the analytic art, is general and comprehenſive, and may be applied with ſucceſs in all caſes where truth is to be obtained and proper data can be eſtabliſhed.

To trace this ſcience to its birth, and to point out the various alterations and improvements it has undergone in its progreſs, would far exceed the limits of a preface. It will be ſufficient to obſerve that the invention is of the higheſt antiquity, and has challenged the praiſe and admiration of all ages. *Diophantus* appears to have been the firſt, among the ancients, who applied it to the ſolution of indeterminate or unlimited problems; but it is to the moderns that we are principally indebted for the moſt curious refinements of the art, and its great and extenſive uſefulneſs in every abſtruſe and difficult inquiry. *Newton*, *Maclaurin*, *Sanderſon*, *Simpſon*, and *Emerſon*, are thoſe, of our own countrymen, who have particularly excelled in this reſpect; and it is to their works that I would refer the young ſtudent, as the paterns of elegance and perfection.

The following compendium is formed entirely upon the model of thoſe writers, and is intended as an uſeful and neceſſary introduction to them. Almoſt every ſubject, which belongs to pure Algebra, is conciſely and diſtinctly treated of; and no pains have been ſpared to make the whole as eaſy and intelligible as poſſible. A great number of elementary books have already been written upon this ſubject; but there are none, which I have yet ſeen, but what appear to me to be extremely defective. Beſides, being totally unfit for the purpoſe of teaching, they are generally calculated to vitiate the taſte, and miſlead the judgment. A tedious and inelegant method prevails through the whole, ſo that the beauty of the ſcience is generally deſtroyed by the clumſy and aukward manner in which it is treated; and the learner, when he is afterwards introduced to ſome of our beſt writers, is obliged to unlearn and forget every thing that he has been at ſo much pains in acquiring.

It is in the ſciences as in every branch of polite literature; there is a certain taſte and elegance which is only to be obtained from the beſt authors, and a judicious uſe of their inſtructions. To direct the ſtudent in his choice of books, and to prepare him properly for the advantages he may receive from them, is, therefore, the buſineſs of every writer who engages in the humble, but uſeful, taſk of a preliminary tutor. This information I have been careful to give, in every part of the preſent performance, where it appeared to be in the leaſt neceſſary; and, though the nature and confined limits of my plan admitted not of diffuſe obſervations, or a formal enumeration of particulars, it is preſumed nothing of real uſe and importance has been omitted. My principal object was to conſult the eaſe, ſatisfaction, and accommodation of the learner; and the favourable reception the work has met with from the public has induced me to give this edition an attentive and careful reviſal.

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ALGEBRA.

DEFINITIONS.

ALGEBRA is the art of computing by ſymbols.

1. *Like quantities* are thoſe that conſiſt of the ſame letters.

2. *Unlike quantities* are thoſe that conſiſt of different letters.

3. *Given quantities* are thoſe whoſe values are known.

4. *Unknown quantities* are thoſe whoſe values are unknown.

5. *Simple quantities* are thoſe that conſiſt of one term only.

6. *Compound quantities* are thoſe that conſiſt of ſeveral terms.

7. *Poſitive* or *affirmative quantities* are thoſe that are to be added.

8. *Negative quantities* are thoſe that are to be ſubtracted.

9. *Like ſigns* are all affirmative (+), or all negative (–).

10. *Unlike ſigns* are when ſome are affirmative (+) and others negative (—).

11. *The co-efficient* of any quantity is the number prefixed to it.

12. *A binomial quantity* is one conſiſting of two terms; a *trinomial* of three terms; a *quadrinomial* of four, &c.

13. *A reſidual quantity* is a binomial where one of the terms is negative.

14. *The power of a quantity* is its ſquare, cube, biquadrate, &c.

15. *The index* or *exponent* is the number which denotes its root or power.

16. *A ſurd,* or *irrational quantity,* is that which has no exact root.

17. *A rational quantity* is that which has no radical ſign ($\surd$), or index annexed to it.

18. *The reciprocal* of any quantity is that quantity inverted, or unity divided by it.

EXPLANATION OF THE CHARCATERS.

$+$	Is the ſign	of addition.
$-$	——	of ſubtraction.
$\times$	——	of multiplication.
$\div$	——	of diviſion.
$:\ ::\ :$		of proportion.
$\surd$	——	of the ſquare root.
$\sqrt[3]{}$	——	of the cube root.
$=$	——	of equality.

Thus $a+b$ ſhews that the number repreſented by b is to be added to that repreſented by a.

$a-b$ ſhews that the number repreſented by b is to be ſubtracted from that repreſented by a.

$a \backsim b$ repreſents the difference of a and b when it is not known which is the greateſt.

ab, or $a\times b$, or $a.b$, denotes the product of the numbers repreſented by a and b.

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$a \div b$ or $\frac{a}{b}$ ſhews that the number repreſented by a is to be divided by the number repreſented by b.

$a : b :: c : d$ denotes that a is in the ſame proportion to b as c is to d.

$x = a - b$ ſhews that x is equal to the difference of a and b.

$\surd a$, or $a^{\frac{1}{2}}$ is the ſquare root of a; $^{3}\surd a$, or $a^{\frac{1}{3}}$, is the cube root of a; and $a^{\frac{1}{m}}$ is the m root of a.

a^2 is the ſquare of a, a^3 is the cube of a; and a^m is the m power of a.

$\frac{a}{b}$ is the reciprocal of the quantity $\frac{b}{a}$, and, $\frac{1}{a}$ is the reciprocal of a.

$\overline{a+b} \times c$, or $(a+b).c$ is the product of the quantity $a+b$ multiplied by the ſimple quantity c.

$\overline{a+b} \div \overline{a-c}$, or $\frac{a+b}{a-c}$, is the quotient of $a+b$ divided by $a-c$.

$\sqrt{ab+cd}$, or $(ab+cd)^{\frac{1}{2}}$ is the ſquare root of the compound quantity $ab+cd$.

$\overline{a+b-c}^{3}$, or $(ab+cd)^3$ is the cube or third power of the quantity $a+b-c$. or $ab+cd$

$5a$ denotes that the quantity a is to be taken 5 times, and $7.(b+c)$ is 7 times $b+c$.

It is alſo to be remarked that the ſign + is generally expreſſed by the word *plus*, or *more*, and the ſign —, by *minus*, or *leſs*.

And, in the computation of problems, it muſt be obſerved, that the firſt letters of the alphabet are uſually put for known quantities, and the laſt for thoſe that are unknown.

ADDITION.

CASE I.

To add quantities which are like, and have like ſigns.

RULE.

Add all the co-efficients together, and to their ſum adjoin the letters common to each term, prefixing the common ſign.

EXAMPLES*.

$$\begin{array}{r} 5\,a \\ 7\,a \\ 8\,a \\ 10\,a \\ 2\,a \\ a \\ \hline 33\,a \\ \hline \end{array} \qquad \begin{array}{r} -\,6\,bx \\ -\,3\,bx \\ -\,2\,bx \\ -\,7\,bx \\ -\,bx \\ -\,5\,bx \\ \hline -\,24\,bx \\ \hline \end{array} \qquad \begin{array}{r} 8\,bxy \\ 7\,bxy \\ 3\,bxy \\ 4\,bxy \\ 5\,bxy \\ bxy \\ \hline 28\,bxy \\ \hline \end{array}$$

$$\begin{array}{r} 2\,y \\ 5\,y \\ 7\,y \\ 4\,y \\ 3\,y \\ \hline 21\,y \\ \hline \end{array} \qquad \begin{array}{r} 5\,x^2+5\,xy \\ 3\,x^2+2\,xy \\ x^2+3\,xy \\ 7\,x^2+8\,xy \\ x^2+2\,xy \\ \hline 17\,x^2+20\,xy \\ \hline \end{array} \qquad \begin{array}{r} 7\,ax-4\,y \\ 8\,ax-3\,y \\ 6\,ax-2\,y \\ 4\,ax-3\,y \\ 2\,ax-2\,y \\ \hline 26\,ax-14\,y \\ \hline \end{array}$$

---

* When a leading quantity has no ſign before it, + is always underſtood; and a quantity without any co-efficient prefixed to it is ſuppoſed to have 1, or unity.

$$\begin{array}{r} 6xy \\ 15xy \\ 2xy \\ 7xy \\ xy \\ xy \\ \hline 30xy \\ \hline \end{array} \qquad \begin{array}{r} -2y^2 \\ -8y^2 \\ -7y^2 \\ -y^2 \\ -6y^2 \\ -y^2 \\ \hline -25y^2 \\ \hline \end{array} \qquad \begin{array}{r} 5a-4b \\ 7a-6b \\ 4a-3b \\ 2a-8b \\ 6a-b \\ 3a-2b \\ \hline 27a-24b \\ \hline \end{array}$$

$$\begin{array}{r} {}^{*}\,20-15x^{\frac{1}{2}}-2xy \\ 35-13x^{\frac{1}{2}}-4xy \\ 18-12x^{\frac{1}{2}}-3xy \\ 12-14x^{\frac{1}{2}}-8xy \\ 10-28x^{\frac{1}{2}}-2xy \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 7xy-5x+3ab \\ 3xy-x+2ab \\ 2xy-3x+2ab \\ 2xy-4x+8ab \\ 5xy-3x+ab \\ \hline \\ \hline \end{array}$$

CASE II.

To add quantities which are like, but have unlike ſigns.

RULE.

1. Add all the affirmative co-efficients into one ſum, and all the negative ones into another.

2. Subtract the leaſt of theſe ſums from the greateſt, and to the difference prefix the ſign of the greateſt, with the common quantity.

---

[*] Quantities, having fractional or other exponents, are to be conſidered, in all reſpects, as if they were expreſſed by a ſingle letter.

EXAMPLES.

$$\begin{array}{r}-3a\\+7a\\+8a\\-a\\-2a\\\hline+9a\\\hline\end{array}\qquad\begin{array}{r}+8ax^2\\+7ax^2\\-3ax^2\\-4ax^2\\+4ax^2\\\hline+12ax^2\\\hline\end{array}\qquad\begin{array}{r}+6x^3+8y\\-3x^3+7y\\-13x^3+8y\\+2x^3-3y\\+x^3-y\\\hline-7x^3+19y\\\hline\end{array}$$

$$\begin{array}{r}-2a^2\\-3a^2\\-8a^2\\+10a^2\\+13a^2\\\hline+10a^2\\\hline\end{array}\qquad\begin{array}{r}+8b^2y^3\\+6b^2y^3\\-10b^2y^3\\-20b^2y^3\\-b^2y^3\\\hline\\\hline\end{array}\qquad\begin{array}{r}-3ab+3\\+8ab-10\\+3ab-6\\-ab+2\\-2ab+11\\\hline\\\hline\end{array}$$

$$\begin{array}{r}-5xy\\-3xy\\+8xy\\+7xy\\\hline\\\hline\end{array}\qquad\begin{array}{r}-8x^2y^2\\+3x^2y^2\\-2x^2y^2\\+4x^2y^2\\\hline\\\hline\end{array}\qquad\begin{array}{r}-12x^2-8x\\+10x^2-3x\\-14x^2+7x\\+8x^2+x\\\hline\\\hline\end{array}$$

$$\begin{array}{r}-2ax^{\frac{1}{2}}\\+ax^{\frac{1}{2}}\\-3ax^{\frac{1}{2}}\\+7ax^{\frac{1}{2}}\\\hline\\\hline\end{array}\qquad\begin{array}{r}-6\sqrt{ax}\\+2\sqrt{ax}\\-5\sqrt{ax}\\+10\sqrt{ax}\\\hline+\sqrt{ax}\\\hline\end{array}\qquad\begin{array}{r}-2y+2ax^{\frac{1}{2}}\\+8y+ax^{\frac{1}{2}}\\-7y-3ax^{\frac{1}{2}}\\+5y+3ax^{\frac{1}{3}}\\\hline\\\hline\end{array}$$

CASE III.

To add quantities which are unlike, and have unlike ſigns.

RULE.

Collect all the like quantities together by the laſt rule, and ſet down thoſe that are unlike, one after another, with their proper ſigns.

EXAMPLES.

$$\begin{array}{r} 5xy \\ 4ax \\ -xy \\ -4ax \\ \hline 4xy \\ \hline \end{array} \qquad \begin{array}{r} 2xy-10x^2 \\ -3x^2+xy \\ -8x^2-xy \\ -xy+9x^2 \\ \hline xy-12x^2 \\ \hline \end{array} \qquad \begin{array}{r} 2ax-150+2\sqrt{x} \\ 3x^2+2ax+6x^2 \\ 5xy-3x^{\frac{1}{2}}+50 \\ \sqrt{x}+100-5x^2 \\ \hline 4ax+4x^2+5xy \\ \hline \end{array}$$

$$\begin{array}{r} 6x^2y^2 \\ -4x^2y \\ -2axy \\ -3x^2y \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 12ax-x^2 \\ 4ax+xy \\ 3y^2-ax \\ 2x^2-24 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 6+10\sqrt{ax}-3y \\ x+4\sqrt{xy}+3y \\ y-2\sqrt{ax}-3y \\ 20+3\sqrt{ax}-3y \\ \hline \\ \hline \end{array}$$

$$\begin{array}{r} 3x^2y \\ -2xy^2 \\ -3y^2x \\ -8x^2y \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 2\sqrt{x}-8y \\ 3\sqrt{xy}+10x \\ 2x+\sqrt{x}+y \\ -8+\sqrt{xy} \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} a^2-8+x^{\frac{1}{2}}-2 \\ a-10+a^2-x \\ x^2-a^2+8-4 \\ 10-a-x^2-y \\ \hline \\ \hline \end{array}$$

SUBTRACTION.

RULE.

Change the ſigns of all the quantities to be ſubtracted, or conceive them to be changed, and then add the different terms together as in addition.

EXAMPLES.

$5a^2-2b$	$6x^2-8y+3$	$5xy-2+8x-y$
$2a^2-5b$	$2x^2+9y-2$	$3xy-8-8x-3y$
$3a^2+3b$	$4x^2-17y+5$	$2xy+6+16x+2y$
$3xy-8$	$2y^2-y-1$	$-10-8x-3xy$
$-xy+8$	y^2+y+1	$xy-7x+3-4ay$
$4xy-16$	y^2-2y-2	$-13-x-4xy+4ay$
$5x^2y-8$	$4\sqrt{xy}-x\sqrt{xy}$	$5x^2y^2+\sqrt{x}-8-4b$
$-3x^2y+1$	$2\sqrt{xy}+2+xy$	$6x^2y^2-10+4b-x^{\frac{1}{2}}$
$8x^2y-9$		$-x^2y^2+\sqrt{x}+2-8b-x^{\frac{1}{2}}$
$3xy-20$	$4x^3-3.(a+b)$	$4xy^3+10a\sqrt{xy+10}$
$4xy-30$	$3x^2-8.(a+b)$	$3x^2y^2+2a\sqrt{xy+10}$

MULTIPLICATION.

CASE I.

When both the factors are ſimple quantities.

RULE.

Multiply the co-efficients of the two terms together, and to the product affix all the letters in thoſe terms, and the reſult will be the whole product required.

*Note**. Like ſigns produce +, and unlike ſigns −.

EXAMPLES.

$$\begin{array}{cccc}
12a & -2a & 5a & -9x \\
3b & +4b & -6x & -5b \\
\hline
36ab & -8ab & -30ax & +45bx \\
\hline
\end{array}$$

$$\begin{array}{cccc}
7ab & 6a^2x & -x^2y & -7xy \\
-5ac & 5x & xy^2 & -xy \\
\hline
-35a^2bc & 30a^2x^2 & -x^3y^3 & +7x^2y^2 \\
\hline
\end{array}$$

$$\begin{array}{cccc}
-5ax & -ax & +5xy & -7xyz \\
3x & -7b & -3 & +6ax \\
\hline
 & & & \\
\hline
\end{array}$$

$$\begin{array}{cccc}
8ax^2 & 5x^2y^2 & -3xy^2 & \surd.(x+y) \\
2ax & x^2y^2 & +2ax & 2\,ax^2 \\
\hline
 & & & \\
\hline
\end{array}$$

---

* That like ſigns make + and unlike ſigns −, in the product, may be ſhewn thus:

1. When $+a$ is to be multiplied by $+b$: this implies that $+a$ is

CASE II.

When one of the factors is a compound quantity.

RULE.

Find the products of the multiplier, and every term of the multiplicand, ſeparately, and place them one after another, with their proper ſigns, and the reſult will be the whole product required.

EXAMPLES.

$$\begin{array}{lll} 4a-2b & 6xy-8 & a^2-2x+6 \\ 3a & 2x & xy \\ \hline 12a^2-6ab & 12x^2y-16x & a^2xy-2x^2y+6xy \\ \hline \end{array}$$

$$\begin{array}{lll} 13x-ab & 35x-7a & 3y-8+2xy \\ 12a & -x & xy \\ \hline & & \\ \hline \end{array}$$

$$\begin{array}{lll} 2x^2+x & 12x^2-4y^2 & 2y^2-8x^2-7x \\ 3xy & -2x^2 & 3xy^2 \\ \hline & & \\ \hline \end{array}$$

---

is to be taken as many times as there are units in *b*; and, ſince the ſum of any number of affirmative terms is affirmative, it is plain that $(+a)\times(+b)=+ab$.

2. If two quantities are to be multiplied together, the reſult will be exactly the ſame, in whatever order they are placed; for *a* times *b* is the ſame as *b* times *a*; and, thereſore, when

CASE III.

When both the factors are compound quantities.

RULE.

Multiply every term of the multiplier into every term of the multiplicand reſpectively, and ſet down the products one after another with their proper ſigns, and their ſum will be the whole product required.

$$\begin{array}{l} x+y \\ x+y \\ \hline x^2+xy \\ \quad +xy+y^2 \\ \hline x^2+2xy+y^2 \\ \hline \end{array} \qquad \begin{array}{l} 5x+4y \\ 3x-2y \\ \hline 15x^2+12xy \\ \quad -10xy-8y^2 \\ \hline 15x^2+2xy-8y^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+xy-y^2 \\ \quad x-y \\ \hline x^3+x^2y-xy^2 \\ \quad -x^2y-xy^2+y^3 \\ \hline x^3 \ * -2xy^2+y^3 \\ \hline \end{array}$$

$$\begin{array}{l} x+y \\ x-y \\ \hline x^2+xy \\ \quad -xy-y^2 \\ \hline x^2 \ * \ -y^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+y \\ x^2+y \\ \hline x^4+yx^2 \\ \quad +yx^2+y^2 \\ \hline x^4+2yx^2+y^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3 \quad * \quad * \quad -y^3 \\ \hline \end{array}$$

---

when $-a$ is to be multiplied by $+b$, or $+b$ by $-a$: this is the ſame as taking $-a$ as many times as there are units in $+b$; and ſince the ſum of any number of negative terms is negative, it is evident that $(-a)\times(+b)$ or $(+b)\times(-a)=-ab$.

3. When $-a$ is to be multiplied by $-b$: Here $a-a=0$, therefore $(a-a)\times -b$ is, alſo $=0$, becauſe 0 multiplied by any quantity produces 0; and ſince the firſt term of the product, or $a\times(-b)$, is (by caſe 2.) $-ab$, the laſt term, or $(-a)\times(-b)$, muſt be $+ab$, in order to make the ſum $(-ab+ab)=0$; conſequently $(-a)\times(-b)=+ab$.

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. *Anſ.* $36a^2x$.
2. Multiply $4x^2-2y$ by $2y$. *Anſ.* $8x^2y-4y^2$.
3. Multiply $2x+4y$ by $2x-4y$. *Anſ.* $4x^2-16y^2$.
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Anſ.* x^4-y^4.
5. Multiply x^2+xy+y^2 by x^2-xy+y^2. *Anſ.* $x^4+x^2y^2+y^4$.
6. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.
7. Multiply $3x^2-2xy+5$ by $x^2+2xy-3$.

DIVISION.

CASE I.

When the diviſor is a ſimple quantity.

RULE.

1. Place the dividend above a ſmall right line, and the diviſor under it, in the manner of a vulgar fraction.

2. Expunge thoſe letters that are common to both the dividend and diviſor, and divide the co-efficients of all the terms by any number that will divide them without a remainder, and the reſult will be the quotient required.

*Note**. Like ſigns make +, and unlike ſigns —, the ſame as in multiplication.

---

* That like ſigns give +, and unlike ſigns — in the quotient, will appear thus:

The diviſor multiplied by the quotient muſt produce the dividend; therefore,

EXAMPLES.

1. It is required to find the quotient of $a \div a$; $8bc \div 2b$; and $abc \div bcd$.

Anſ. $\frac{a}{a}=1$; $\frac{8bc}{2b}=4c$; *and* $\frac{abc}{bcd}=\frac{a}{d}$

2. It is required to find the quotient of $12xy \div 6x^2$, and, $(ab+b^2) \div 2b$.

Anſ. $\frac{12xy}{6x^2}=\frac{2y}{x}$, *and* $\frac{ab+b^2}{2b}=\frac{a+b}{2}$

3. Divide $18x^2$ by $9x$. *Anſ.* $2x$.
4. Divide $10x^2y^2$ by $-5x^2y$. *Anſ.* $-2y$.
5. Divide $-9ax^2y^2$ by $9x^2y$. *Anſ.* $-ay$.
6. Divide $-8x^2$ by $-2x$. *Anſ.* $+4x$.
7. Divide $10ab+15ac$ by $5a$. *Anſ.* $2b+3c$.
8. Divide $30ax-54x$ by $6x$. *Anſ.* $5a-9$.
9. Divide $10x^2y-15y^2-5y$ by $5y$.
Anſ. $2x^2-3y-1$.
10. Divide $a+3ax-x^2$ by a.
11. Divide $3a^2-15+6a+3b$ by $3a$.

CASE II.

When the diviſor and dividend are both compound quantities.

RULE.

1. Range the terms of both the quantities according to the dimenſions of ſome letter in them; ſo that the

---

1. When both the terms are +, the quotient is +, becauſe (+)×(+) produces + in the dividend.

2. When they are both —, the quotient is alſo+, becauſe (+)×(—) produces — in the dividend.

3. When one of them is + and the other —, the quotient is —, becauſe (—)×(+) produces — in the dividend; and (—) ×(—) produces + in the dividend.

the firſt term may contain the higheſt power of that letter; the ſecond term, the next higheſt power; and ſo on.

2. Divide the firſt term of the dividend by the firſt term of the diviſor, and place the reſult in the quotient.

3. Multiply the whole diviſor by the term thus found, and ſubtract the reſult from the dividend.

4. To this remainder bring down as many terms of the dividend as are requiſite for the next operation, and divide as before; and ſo on, as in common arithmetic.

Note. If the diviſor be not exactly contained in the dividend, the quantity which remains after the operation is finiſhed, muſt be placed over the diviſor, like a vulgar fraction, and ſet down at the end of the quotient, as in common arithmetic.

EXAMPLES.

$$
\begin{array}{l}
x+y)\,x^2+2xy+y^2\,(x+y \\
\underline{x^2+xy} \\
xy+y^2 \\
\underline{xy+y^2} \\
*
\end{array}
$$

$$
\begin{array}{l}
a+x)\,a^3+5a^2x+5ax^2+x^3\,(a^2+4ax+x^2 \\
\underline{a^3+a^2x} \\
4a^2x+5ax^2 \\
\underline{4a^2x+4ax^2} \\
ax^2+x^3 \\
\underline{ax^2+x^3} \\
*
\end{array}
$$

$$x-3)\,x^3-9x^2+27x-27\,(x^2-6x+9$$

$$x^3-3x^2$$

$$-6x^2+27x$$

$$-6x^2+18x$$

$$9x-27$$

$$9x-27$$

*

$$a-x)\,a^3-x^3\,(a^2+ax+x^2$$

$$a^3-a^2x$$

$$a^2x-x^3$$

$$a^2x-ax^2$$

$$ax^2-x^3$$

$$ax^2-x^3$$

*

$$b-y)\,b^4-y^4\,(b^3+b^2y+by^2+y^3$$

$$b^4-b^3y$$

$$b^3y-y^4$$

$$b^3y-b^2y^2$$

$$b^2y^2-y^4$$

$$b^2y^2-by^3$$

$$by^3-y^4$$

$$by^3-y^4$$

*

EXAMPLES FOR PRACTICE:

1. Divide $a^2+2ax+x^2$ by $a+x$. *Anſ.* $a+x$.
2. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$. *Anſ.* $a^2-2ay+y^2$.
3. Divide 1 by $1-x$. *Anſ.* $1+x+x^2+x^3$, &c.
4. Divide $6x^4-96$ by $3x-6$. *Anſ.* $2x^3+4x^2+8x+16$.
5. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$. *Anſ.* $a^3-3a^2x+3ax^2-x^3$.
6. Divide $3x^3-12x^2-ax^2+10ax-2a^2$ by $3x-a$.
7. Divide $y^6-3y^4x^2+3y^2x^4-x^6$ by $y^3-3y^2x+3yx^2-x^3$.

ALGEBRAIC FRACTIONS.

PROBLEM I.

To reduce a mixed quantity to an improper fraction.

RULE.

Multiply the integer by the denominator of the fraction, and to the product add the numerator; and the denominator being placed under this ſum will give the improper fraction required.

EXAMPLES:

1. Let $3\frac{5}{7}$ and $a-\frac{b}{c}$ be reduced to improper fractions.

Firſt, $3\frac{5}{7}=\frac{3\times7+5}{7}=\frac{21+5}{7}=\frac{26}{7}$ *Anſ.*

And, $a-\frac{b}{c}=\frac{a\times c-b}{c}=\frac{ac-b}{c}$ *Anſ.*

2. Let $x+\frac{x^2}{a}$ and $x-\frac{a^2-x^2}{x}$ be reduced to improper fractions.

Firſt, $x+\frac{x^2}{a}=\frac{x\times a+x^2}{a}=\frac{ax+x^2}{a}$ *the Anſwer.*

And, $x-\frac{a^2-x^2}{x}=\frac{x^2-a^2+x^2}{x}=-\frac{a^2}{x}$ *Anſ.*

3. Reduce $8\frac{6}{7}$ to an improper fraction. *Anſ.* $\frac{62}{7}$

4. Reduce $1-\frac{2x}{a}$ to an improper fraction. *Anſ.* $\frac{a-2x}{a}$

5. Let $x-\frac{ax+x^2}{2a}$ be reduced to an improper fraction.

6. Let $10+\frac{2x-8}{3x}$ be reduced to an improper fraction.

7. Let $a+\frac{1-x-a}{b}$ be reduced to an improper fraction.

8. Let $1+2x-\frac{x-3}{5x}$ be reduced to an improper fraction.

PROBLEM II.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator, for the integral part, and place the remainder, if any, over the denominator, and it will be the mixed quantity required.

EXAMPLES:

1. Let $\frac{17}{5}$ and $\frac{ax+a^2}{x}$ be reduced to whole or mixed quantities.

Firſt, $\frac{17}{5}=17\div5=3\frac{2}{5}$ *the Anſwer required.*

And, $\frac{ax+a^2}{x}=(ax+a^2)\div x=a+\frac{a^2}{x}$ *Anſ.*

2. Let $\frac{ab-a^2}{b}$ and $\frac{ay+2y^2}{a+y}$ be reduced to whole or mixed quantities.

Firſt, $\frac{ab-a^2}{b}=(ab-a^2)\div b=a-\frac{a^2}{b}$ *Anſ.*

And, $\frac{ay+2y^2}{a+y}=(ay+2y^2)\div(a+y)=y+\frac{y^2}{a+y}$

3. Let $\frac{35}{8}$ and $\frac{3ab-b^2}{a}$ be reduced to whole or mixed quantities. *Anſ.* $4\frac{3}{8}$ *and* $3b-\frac{b^2}{a}$

4. Let $\frac{2x^2y}{2x}$ and $\frac{a^2+x^2}{a-x}$ be reduced to whole or mixed quantities.

5. Let $\frac{x^2-y^2}{x+y}$ and $\frac{x^3-y^3}{x-y}$ be reduced to whole or mixed quantities.

6. Let $\frac{10x^2-x+3}{5x}$ be reduced to a whole or mixed quantity.

7. Let $\frac{12x^3+3x^2}{4x^3+x^2-4x-1}$ be reduced to a whole or mixed quantity.

PROBLEM III.

To reduce fractions of different denominators, to others of the ſame value, which ſhall have a common denominator.

RULE.

Multiply every numerator, ſeparately, into all the denominators but its own, for the new numerators, and all the denominators together for the common denominator *.

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to fractions of equal values that ſhall have a common denominator.

$$\begin{array}{c} a \times c = ac \\ b \times b = b^2 \\ \hline b \times c = bc \end{array}$$

$$\frac{ac}{bc} \text{ and } \frac{b^2}{bc} = \textit{fractions required.}$$

2. Reduce $\frac{a}{b}$, $\frac{b}{c}$ and $\frac{c}{d}$ to equivalent fractions having a common denominator.

$$\begin{array}{c} a \times c \times d = acd \\ b \times b \times d = b^2 d \\ c \times b \times c = c^2 b \\ \hline b \times c \times d = bcd \end{array}$$

$$\frac{acd}{bcd}, \frac{b^2 d}{bcd}, \text{ and } \frac{c^2 b}{bcd} = \textit{fractions required.}$$

---

* When the denominators have a common diviſor, it will be better, inſtead of multiplying by the whole denominators, to multiply only by thoſe parts which ariſe from dividing by the common diviſor.

3. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator. *Anſ.* $\frac{2xc}{ac}$ *and* $\frac{ab}{ac}$

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to fractions, having a common denominator. *Anſ.* $\frac{ac}{bc}$ *and* $\frac{ab+b^2}{bc}$

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d to fractions, having a common denominator. *Anſ.* $\frac{9xc}{6ac}$, $\frac{4ab}{6ac}$, *and* $\frac{6adc}{6ac}$.

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{2x}{a}$ to fractions, having a common denominator.

Anſ. $\frac{9a}{12a}$, $\frac{8ax}{12a}$, *and* $\frac{12a^2+24x}{12a}$.

7. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$, and $\frac{x^2+2^2}{a+a}$ to fractions, having a common denominator.

8. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$ and $\frac{d}{a}$ to equivalent fractions, having a common denominator.

PROBLEM IV.

To find the greateſt common meaſure of a fraction.

RULE *.

1. Range the quantities according to the dimenſions of ſome letter, as is ſhewn in diviſion.

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* The ſimple diviſors, in this rule, may be eaſily found, by inſpection.

2. Divide the greater term by the leſs, and the laſt diviſor by the laſt remainder, and ſo on till nothing remains; and the diviſor laſt uſed will be the common meaſure required.

Note. All the letters or figures which are common to each diviſor, muſt be thrown out of them before they are uſed in the operation.

EXAMPLES.

1. To find the greateſt common meaſure of $\frac{cx+x^2}{ca^2+a^2x}$

$$
\begin{array}{rl}
cx+x^2) & ca^2+a^2x \\
\text{or } c+x\ ) & ca^2+a^2x\,(a^2 \\
 & ca^2+a^2x \\
\hline
 & *
\end{array}
$$

Therefore the greateſt common meaſure is $c+x$.

2. To find the greateſt common meaſure of $\frac{x^3-b^2x}{x^2+2bx+b^2}$

$$
\begin{array}{rl}
x^2+2bx+b^2) & x^3-b^2x\,(x \\
 & x^3+2bx^2+b^2x \\
\hline
-2bx^2-2b^2x) & x^2+2bx+b^2 \\
\text{or } x+b\ ) & x^2+2bx+b^2\,(x+b \\
 & x^2+bx \\
\hline
 & bx+b^2 \\
 & bx+b^2 \\
\hline
 & *
\end{array}
$$

Therefore $x+b$ *is the greateſt common diviſor.*

3. To find the greateſt common diviſor of $\frac{x^2-1}{xy+y}$. *Anſ.* $x+1$.

4. To find the greateſt common diviſor of $\frac{x^4-b^4}{x^5-b^2x^3}$. *Anſ.* x^2-b^2.

5. To find the greateſt common meaſure of $\frac{5a^5+10a^4b+5a\,b^2}{a^3b+2a^2b^2+2ab^3+b^4}$.

PROBLEM V.

To reduce a fraction to its loweſt terms.

RULE.

1. Find the greateſt common meaſure, as in the laſt problem.

2. Divide both the terms of the fraction by the common meaſure thus found, and it will reduce it to its loweſt terms as was required.

EXAMPLES.

1. Reduce $\frac{cx+x^2}{ca^2+a^2x}$ to its loweſt terms.

$$\begin{array}{rl} cx+x^2) & ca^2+a^2x \\ \text{or } c+x\,) & ca^2+a^2x\,(a^2 \\ & ca^2+a^2x \\ & \overline{\qquad\quad *\qquad\quad} \end{array}$$

Here $cx+x^2$ *is divided by* x *which is common to each term.*

Therefore $c+x$ is the greateſt common meaſure,

$$\textit{and } c+x)\frac{cx+x^2}{ca^2+a^2x}=\frac{x}{a^2}=\textit{ fraction required.}$$

2. Having $\frac{x^3-b^2x}{x^2+2bx+b}$ given, it is required to reduce it to its leaſt terms.

$$\begin{array}{r}
x^2+2bx+b^2)x^3-b^2x(x \\
x^3+2bx^2+b^2x \\
\hline
-2bx^2-2b^2x)x^2+2bx+b^2 \\
\textit{or } x+b)x^2+2bx+b^2(x+b \\
x^2+bx \\
\hline
bx+b^2 \\
bx+b^2 \\
\hline
*
\end{array}$$

Therefore $x+b$ is the greateſt common meaſure,

$$\textit{and } x+b)\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x^2-bx}{x+b}=\textit{ fraction required.}$$

3. Reduce $\frac{x^4-b^4}{x^5-b^2x^3}$ to its loweſt terms. *Anſ.* $\frac{x^2+b^2}{x^3}$

4. Reduce $\frac{x^2-y^2}{x^4-y^4}$ to its loweſt terms. *Anſ.* $\frac{1}{x^2+y^2}$

5. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its loweſt terms.

6. Reduce $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$ to its loweſt terms.

PROBLEM VI.

To add fractional quantities together.

RULE.

1. Reduce the fractions to a common denominator, as in problem the third.

2. Add all the numerators together, and under their ſum write the common denominator, and it will give the ſum of the fractions required.

EXAMPLES.

1. Having $\frac{x}{2}$ and $\frac{x}{3}$ given, to find their ſum.

$$\begin{array}{c} x \times 3 = 3x \\ x \times 2 = 2x \\ \hline 2 \times 3 = 6 \end{array}$$

$\frac{3x}{6}+\frac{2x}{6}=\frac{5x}{6}=$ *ſum required.*

2. Having $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ given, to find their ſum.

$$\begin{array}{c} a \times d \times f = adf \\ c \times b \times f = cbf \\ e \times b \times d = ebd \\ \hline b \times d \times f = bdf \end{array}$$

$\frac{adf}{bdf}+\frac{cbf}{bdf}+\frac{ebd}{bdf}=\frac{adf+cbf+ebd}{bdf}=$ *ſum required.*

*3. Let $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$ be added together.

$$\left.\begin{array}{l} 3x^2\times c = 3cx^2 \\ 2ax\times b = 2abx \end{array}\right\} = \textit{numerators.}$$

And $b\times c = bc =$ *common denominator.*

Therefore $a-\frac{3x^2}{b}+b+\frac{2ax}{c}=a-\frac{3cx^2}{bc}+b+\frac{2abx}{bc}=$ $a+b+\frac{2abx-3cx^2}{bc}=$ *ſum required.*

4. Add $\frac{3x}{2b}$ and $\frac{x}{5}$ together. *Anſ.* $\frac{15x+2bx}{10b}$

5. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together. *Anſ.* $x+\frac{x}{12}$

6. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. *Anſ.* $\frac{19x-14}{21}$

7. Add $x+\frac{x-2}{3}$ to $3x+\frac{2x-3}{4}$. *Anſ.* $4x+\frac{10x-17}{12}$.

8. It is required to add $4x$, $\frac{5x^2}{2a}$ and $\frac{x+a}{2x}$ together.

9. It is required to add $\frac{2x}{3}$, $\frac{7x}{4}$ and $\frac{2x+1}{5}$ together.

10. It is required to add $4x$, $\frac{7x}{9}$ and $2+\frac{x}{5}$ together.

11. It is required to add $3x+\frac{2x}{5}$ and $x-\frac{8x}{9}$ together.

---

[*] In the addition of mixed quantities, it is beſt to bring the fractional parts only to a common denominator, and to affix their ſum to the ſum of the integers.

PROBLEM VII.

To ſubtract one fractional quantity from another.

RULE.

1. Reduce the fractions to a common denominator, as in addition.

2. Subtract the numerators from each other, and under their difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES.

1. To find the difference of $\frac{x}{3}$ and $\frac{2x}{11}$.

$$x\times 11=11x$$
$$2x\times 3=6x$$

$$3\times 11=33$$

$$\frac{11x}{33}-\frac{6x}{33}=\frac{5x}{33}=\text{ difference required.}$$

2. To find the difference of $\frac{x-a}{3b}$ and $\frac{2a-4x}{5c}$.

$$(x-a)\times 5c=5cx-5ac$$
$$(2a-4x)\times 3b=6ab-12bx$$

$$3b\times 5c=15bc$$

$$\frac{5cx-5ac}{15bc}-\frac{6ab-12bx}{15bc}=\frac{5cx-5ac-6ab+12bx}{15bc}=$$
difference required.

---

[*] The ſame rule may be obſerved for mixed quantities in ſubtraction as in addition.

3. Find the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$ *Anſ.* $\frac{39x}{35}$

4. Find the difference of $5y$ and $\frac{3y}{8}$ *Anſ.* $\frac{37y}{8}$

5. Find the difference of $\frac{3x}{7}$ and $\frac{2x}{9}$ *Anſ.* $\frac{13x}{63}$

6. Required the difference between $\frac{x+a}{b}$ and $\frac{c}{d}$.

Anſ. $\frac{dx+ad-bc}{bd}$

7. Required the difference of $\frac{3x+a}{5b}$ and $\frac{2x+7}{8}$.

Anſ. $\frac{24x+8a-10bx-35b}{40b}$

8. Find the difference of $3x+\frac{x}{b}$ and $x+\frac{x-a}{c}$

Anſ. $2x+\frac{cx-bx+ab}{bc}$

PROBLEM VIII.

To multiply fractional quantities together.

RULE [*].

Multiply the numerators together for a new numerator, and the denominators for a new denominator, and it will give the product required.

---

[*] 1. When the numerator of one fraction, and the denominator of the other, can be divided by ſome quantity which is common to each, the quotients may be uſed inſtead of them.

2. When

EXAMPLES.

1. Let it be required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

$$\left.\begin{matrix} x\times 2x \\ 6\times 9 \end{matrix}\right\} = \frac{2x^2}{54} = \frac{x^2}{27} = \textit{product required.}$$

2. Find the product of $\frac{x}{2}$, $\frac{4x}{5}$ and $\frac{10x}{21}$.

$$\left.\begin{matrix} x\times 4x\times 10x \\ 2\times 5\ \times 21 \end{matrix}\right\} = \frac{40x^3}{210} = \frac{4x^3}{21} = \textit{product required.}$$

3. Find the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$.

$$\left.\begin{matrix} x\times (x+a) \\ a\times (a+c) \end{matrix}\right\} = \frac{x^2+ax}{a^2+ac} = \textit{product required.}$$

4. Find the product of $\frac{3x}{2}$ and $\frac{3a}{b}$ *Anſ.* $\frac{9ax}{2b}$

5. Find the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$ *Anſ.* $\frac{3x^3}{5a}$

6. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$ and $\frac{3ac}{2b}$.

Anſ. $9ax$.

7. It is required to find the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$.

Anſ. $\frac{ab+bx}{x}$.

---

2. When a fraction is to be multiplied by an integer, the product is found by multiplying it by the numerator; and if the integer be the ſame with the denominator, the numerator may be taken for the product.

3. When a fraction is to be multiplied by any quantity, it is the ſame thing whether the numerator be multiplied by it, or the denominator divided by it.

8. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

9. Required the product of x, $\frac{x+1}{a}$ and $\frac{x-1}{a+b}$.

PROBLEM IX.

To divide one fractional quantity by another.

RULE *.

Multiply the denominator of the diviſor by the numerator of the dividend, for a new numerator, and the numerator of the diviſor by the denominator of the dividend, for a new denominator.

Or, which is the ſame thing, invert the diviſor, and proceed exactly as in multiplication.

EXAMPLES.

1. Find the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$.

$$\frac{x}{3}\times\frac{9}{2x}=\frac{9x}{6x}=\frac{3}{2}=1\tfrac{1}{2}=\textit{quotient required.}$$

2. Find the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$.

$$\frac{2a}{b}\times\frac{d}{4c}=\frac{2ad}{4bc}=\frac{ad}{2bc}=\textit{quotient required.}$$

---

* 1. If the fractions to be divided have a common denominator, take the numerator of the dividend for a new numerator, and the numerator of the diviſor for the denominator.

3. Find the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

$\frac{x+a}{2x-2b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{2x^2-2b^2} =$ *quotient required.*

4. Find the quotient of $\frac{2x^2}{a^3+x^3}$ divided by $\frac{x}{x+a}$.

$\frac{2x^2}{a^3+x^3} \times \frac{x+a}{x} = \frac{2x^2 \times (x+a)}{(a^3+x^3) \times x} = \frac{2x}{x^2-ax+a^2} =$ *quotient required.*

5. Let $\frac{7x}{5}$ be divided by $\frac{12}{13}$. Anſ. $\frac{91x}{60}$.

6. Let $\frac{4x^2}{7}$ be divided by $5x$. Anſ. $\frac{4x}{35}$.

7. Let $\frac{x+1}{6}$ be divided by $\frac{2x}{3}$ Anſ. $\frac{x+1}{4x}$.

8. Let $\frac{x}{x-1}$ be divided by $\frac{x}{2}$. Anſ. $\frac{2}{x-1}$.

9. Let $\frac{5x}{3}$ be divided by $\frac{2a}{3b}$. Anſ. $\frac{5bx}{2a}$.

10. Let $\frac{x-b}{8cd}$ be divided by $\frac{3cx}{4d}$ Anſ. $\frac{x-b}{6c^2x}$.

11. Let $\frac{x^4-b^4}{x^2-2bx+b^2}$ be divided by $\frac{x^2+bx}{x-b}$.

Anſ. $x+\frac{b^2}{x}$.

---

2. When a fraction is to be divided by any quantity, it is the ſame thing whether the numerator be divided by it, or the denominator multiplied by it.

3. When the two numerators, or the two denominators, can be divided by ſome common quantity, that quantity may be thrown out of each, and the quotients uſed inſtead of the fractions firſt propoſed.

INVOLUTION.

Involution is the raiſing of powers from any propoſed root; or the method of finding the ſquare, cube, biquadrate, &c. of any given quantity.

RULE*.

Multiply the quantity into itſelf as many times as there are units in the index leſs one, and the laſt product will be the power required. *Or*

Multiply the index of the quantity by the index of the power, and the reſult will be the ſame as before.

Note, When the ſign of the root is + all the powers of it will be +; and when the ſign is — all the odd powers will be —, and all the even powers +.

EXAMPLES.

$$a,\ \textit{root}\left\{\begin{array}{l} a^2 = \textit{ſquare} \\ a^3 = \textit{cube} \\ a^4 = 4\textit{th power} \\ a^5 = 5\textit{th power} \end{array}\right. \qquad a^2,\ \textit{root}\left\{\begin{array}{l} a^4 = \textit{ſquare} \\ a^6 = \textit{cube} \\ a^8 = 4\textit{th power} \\ a^{10} = 5\textit{th power} \end{array}\right.$$

&c. &c.

$$-3a,\ \textit{root}\left\{\begin{array}{l} +\ \ 9a^2 = \textit{ſquare} \\ -\ \ 27a^3 = \textit{cube} \\ +\ \ 81a^4 = 4\textit{th power} \\ -243a^5 = 5\textit{th power} \end{array}\right.$$

---

* The *n*th. power of any product is equal to the *n*th. power of each of the factors, multiplied together.

And the *n*th. power of a fraction, is equal to the *n*th. power of the numerator, divided by the *n*th. power of the denominator.

$$-2ax^2, \textit{ root} \left\{ \begin{array}{l} +\ 4a^2x^4 = \textit{ſquare} \\ -\ 8a^3x^6 = \textit{cube} \\ +16a^4x^8 = \textit{4th power} \\ -32a^5x^{10} = \textit{5th power} \end{array} \right.$$

$$\frac{x}{a}, \textit{ root} \left\{ \begin{array}{l} \dfrac{x^2}{a^2} = \textit{ſquare} \\ \dfrac{x^3}{a^3} = \textit{cube} \\ \dfrac{x^4}{a^4} = \textit{biquadrate} \end{array} \right.$$

$$-\frac{2ax^2}{3b}, \textit{ root} \left\{ \begin{array}{l} +\ \dfrac{4a^2x^4}{9b^2} = \textit{ſquare} \\ -\ \dfrac{8a^3x^6}{27b^3} = \textit{cube} \\ +\ \dfrac{16a^4x^8}{81b^4} = \textit{4th power} \end{array} \right.$$

$$\begin{array}{lllll} x + a = \textit{root} & & & & \\ x + a & & & & \\ \hline x^2 + ax & & & & \\ \quad + ax + a^2 & & & & \\ \hline x^2 + 2ax + a^2 = \textit{ſquare} & & & & \\ x + a & & & & \\ \hline x^3 + 2ax^2 + a^2x & & & & \\ \quad + ax^2 + 2a^2x + a^3 & & & & \\ \hline x^3 + 3ax^2 + 3a^2x + a^3 = \textit{cube} & & & & \\ \hline \end{array}$$

or third power of $x + a$.

EXAMPLES FOR PRACTICE.

1. Required the cube or third power of $2a^2$.
Anſ. $8a^6$.

2. Required the 4th power of $2a^2x$. *Anſ.* $16a^8x^4$.

3. Required the 3d. power of $-8x^2y^3$.
Anſ. $-512x^6y^9$.

4. To find the biquadrate of $-\frac{2a^2x}{3b}$. *Anſ.* $\frac{16a^8x^4}{81b^8}$.

5. Required the 5th power of $a-x$.
Anſ. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

SIR ISAAC NEWTON's RULE *for raiſing a binomial or reſidual quantity to any power whatever* *.

1. *To find the terms without the co-efficients.* The index of the firſt, or leading quantity, begins with that of the given power, and decreaſes continually by 1, in every term to the laſt; and in the following quantity the indices of the terms are 0, 1, 2, 3, 4, &c.

2. *To find the unciæ or co-efficients.* The firſt is always 1, and the ſecond is the index of the power: and in general, if the co-efficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the co-efficient of the term next following.

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* This rule, expreſſed in general terms, is as follows:

$$\overline{a+b}^n = a^n + n.a^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 \text{ \&c.}$$

$$\overline{a-b}^n = a^n - n.a^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 - n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 \text{ \&c.}$$

Note, The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the ſecond term be —, all the odd terms will be +, and the even terms —.

EXAMPLES.

1. Let $a+x$ be involved to the fifth power.

The terms without the co-efficients will be

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5,$$

and the co-efficients will be

$$1,\ 5,\ \frac{5\times4}{2},\ \frac{10\times3}{3},\ \frac{10\times2}{4},\ \frac{5\times1}{5};$$

And therefore the 5th power is

$$a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

2. Let $x-a$ be involved to the 6th power.

The terms without the co-efficients will be

$$x^6,\ x^5a,\ x^4a^2,\ x^3a^3,\ x^2a^4,\ xa^5,\ a^6,$$

and the co-efficients will be

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6},$$

or 1, 6, 15, 20, 15, 6, 1,

And therefore the 6th power of $x-a$ *is*

$$x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6.$$

3. Find the 4th power of $x-a$.

Anſ. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.

4. Find the 7th power of $x+a$.

Anſ. $x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7$.

The ſum of the co-efficients in every power is equal to the number 2, raiſed to that power. This $1+1=2$ for the ſquare; $1+2+1=4=2^2$ for the cube; $1+3+3+1=8$ for the 4th power, &c.

EVOLUTION.

Evolution is the reverſe of involution, or the method of finding the ſquare root, cube root, &c. of any given quantity.

CASE I.

To find the roots of ſimple quantities.

RULE *.

Extract the root of the co-efficient for the numerical part, and divide the index of the letter, or letters, by the index of the power, and it will give the root required.

EXAMPLES.

1. Required the ſquare root of $9x^2$, and the cube root of $8x^3$.

Anſ. $\sqrt{9x^2}=3x^{\frac{2}{2}}=3x$; *and* $\sqrt[3]{8x^3}=2x^{\frac{3}{3}}=2x$.

2. Required the ſquare root of $\frac{3x^2y^2}{4a^2}$, and the cube root of $\frac{16x^4y^3}{27a^3}$.

Anſ. $\sqrt{\frac{3x^2y^2}{4a^2}}=\frac{xy}{2a}\sqrt{3}$; *and* $\sqrt[3]{\frac{16x^4y^3}{27a^3}}=\frac{2xy}{3a}.\sqrt{2x}$.

---

* Any even root of an affirmative quantity may be either + or −; thus the ſquare root of $+a^2$ is either $+a$, or $-a$: for $(+a)\times(+a)=+a^2$, and $(-a)\times(-a)=+a^2$.

And an odd root of any quantity will have the ſame ſign as the quantity itſelf: thus the cube root of $+a^3$ is $+a$, and the cube

3. Required the ſquare root of $3a^2x^6$. *Anſ.* $ax^3\sqrt{3}$.

4. Required the cube root of $-125a^3x^6$. *Anſ.* $-5ax^2$.

5. Required the ſquare root of $\frac{x^2y^2}{a^3}$. *Anſ.* $\frac{xy}{a}\times\frac{1}{\sqrt{a}}$.

6. Required the 4th root of $16a^4x^8$. *Anſ.* $2ax^2$.

7. It is required to find the 5th root of $-32x^5y^{10}$.
Anſ. $-2xy^2$.

CASE II.

To find the ſquare root of a compound quantity.

RULE.

1. Range the quantities according to the dimenſions of ſome letter, and ſet the root of the firſt term in the quotient.

2. Subtract the ſquare of the root, thus found, from the firſt term, and bring down the two next terms to the remainder for a dividend.

3. Divide the dividend by double the root, and ſet the reſult both in the quotient and diviſor.

4. Multiply the diviſor, thus increaſed, by the term laſt put in the quotient, and ſubtract the product from the dividend; and ſo on, as in common arithmetic.

---

cube root of $-a^3$ is $-a$; for $(+a)\times(+a)\times(+a)=+a^3$, and $(-a)\times(-a)\times(-a)=-a^3$.

Any even root of a negative quantity is impoſſible; for neither $(+a)\times(+a)$ nor $(-a)\times(-a)$ can produce $-a^2$.

† The *n*th. root of a product is equal to the *n*th. root of each of the factors multiplied together.

And the *n*th. root of a fraction is equal to the *n*th. root of the numerator, divided by the *n*th. root of the denominator.

EXAMPLES.

1. Extract the ſquare root of $x^4-4x^3+6x^2-4x+1$.

$$\begin{array}{r l}
 & x^4-4x^3+6x^2-4x+1\,(x^2-2x+1=\textit{root}\\
 & x^4\\ \cline{2-2}
2x^2-2x) & -4x^3+6x^2\\
 & -4x^3+4x^2\\ \cline{2-2}
2x^2-4x+1) & 2x^2-4x+1\\
 & 2x^2-4x+1\\ \cline{2-2}
 & *
\end{array}$$

2. Extract the ſquare root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

$$\begin{array}{r l}
 & 4a^4+12a^3x+13a^2x^2+6ax^3+x^4\,(2a^2+3ax+x^2\\
 & 4a^4\\ \cline{2-2}
4a^2+3ax) & 12a^3x+13a^2x^2\\
 & 12a^3x+\ 9a^2x^2\\ \cline{2-2}
4a^2+6ax+x^2) & 4a^2x^2+6ax^3+x^4\\
 & 4a^2x^2+6ax^3+x^4\\ \cline{2-2}
 & *
\end{array}$$

3. Required the ſquare root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.
Anſ. $a^2+2ax+x^2$.

4. Required the ſquare root of $x^4-2x^3+\frac{3}{2}x^2-\frac{x}{2}+\frac{1}{16}$.
Anſ. $x^2-x+\frac{1}{4}$.

5. It is required to find the ſquare root of a^2+x^2.

Anſ. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$ &c.

CASE III.

To find the roots of powers in general.

RULE *.

1. Find the root of the firſt term, and place it in the quotient.

2. Subtract its power, and bring down the ſecond term for a dividend.

3. Involve the root, laſt found, to the next loweſt power, and multiply it by the index of the given power for a diviſor.

4. Divide the dividend by the diviſor, and the quotient will be the next term of the root.

5. Involve the whole root, and ſubtract and divide as before; and ſo on till the whole is finiſhed.

EXAMPLES.

1. Required the ſquare root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$\begin{array}{l}
a^4-2a^3x+3a^2x^2-2ax^3+x^4(a^2-ax+x^2 \\
a^4 \\
\hline
2a^2)-2a^3x \\
\hline
a^4-2a^3x+a^2x^2 \\
\hline
\quad 2a^2)2a^2x^2 \\
\hline
a^4-2a^3x+3a^2x^2-2ax^3+x^4 \\
\hline
\qquad\qquad *
\end{array}$$

---

* As this method, in high powers, is generally thought too laborious, it may not be improper to obſerve that the roots of compound quantities may ſometimes be eaſily diſcovered thus:

1. Extract

2. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$\begin{array}{l} x^6+6x^5-40x^3+96x-64(x^2+2x-4 \\ x^6 \\ \hline 3x^4)\ 6x^5 \\ \hline x^6+6x^5+12x^4+8x^3 \\ \hline \quad 3x^4)-12x^4 \\ \hline x^6+6x^5-40x^3+96x-64 \\ \hline \end{array}$$

*

3. Required the ſquare root of $a^2+2ab+2ac+b^2+2bc+c^2$. *Anſ.* $a+b+c$.

4. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. *Anſ.* x^2-2x+1.

5. Required the biquadrate root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$. *Anſ.* $2a-3x$.

6. Required the fifth root of $32x^5-80x^4+80x^3-40x^2+10x-1$. *Anſ.* $2x-1$.

---

1. Extract the roots of all the ſimple terms, and connect them together by the ſign + or —, as may be judged ſuitable for the purpoſe.

2. Involve the compound root, thus found, to the proper power, and, if it be the ſame with the given quantity, it is the root required.

3. But if it be found to differ only in ſome of the ſigns, change them from + to —, or from — to +, till its power agrees with the given one throughout.

Thus, in the fifth example, the root $2a-3x$ is the difference of the roots of the firſt and laſt terms; and in the 3d example the root $a+b+c$ is the ſum of the roots of the 1ſt, 4th and 6th terms. The ſame may alſo be obſerved of the 6th example; where the root is found from the firſt and laſt terms.

S U R D S.

Surds are ſuch quantities as have no exact root, being uſually expreſſed by fractional indices, or by means of the radical ſign $\surd$.

Thus, $2^{\frac{1}{2}}$ denotes the ſquare root of 2, and $3^{\frac{2}{3}}$ the cube root of the ſquare of 3; where the numerator ſhews the power to which the quantity is to be raiſed, and the denominator its root.

PROBLEM I.

To reduce a rational quantity to the form of a Surd.

RULE.

Raiſe the quantity to a power equivalent to that denoted by the index of the ſurd, and over this new quantity place the radical ſign, and it will be of the form required.

EXAMPLES.

1. It is required to reduce 3 to the form of the ſquare root.

Firſt $3\times3=3^2=9$; *whence* $\sqrt{9}$ *the anſwer.*

2. It is required to reduce $2x^2$ to the form of the cube root.

Firſt, $2x^2\times2x^2\times2x^2=(2x^2)^3=8x^6$; *whence* $\sqrt[3]{8x^6}$ *or* $(8x^6)^{\frac{1}{3}}$ *the anſwer.*

3. Reduce 5 to the form of the cube root.

Anſ. $(125)^{\frac{1}{3}}$

4. Reduce $\frac{1}{2}xy$ to the form of the ſquare root.

Anſ. $\sqrt{\frac{1}{4}x^2y^2}$.

5. Reduce 2 to the form of the 5th root. *Anſ.* $(32)^{\frac{1}{5}}$.

6. Let $2^{\frac{1}{2}}$ be reduced to the form of the 7th root.

7. Reduce $a+b$ to the form of the ſquare root, and $a-b$ to the form of the cube root.

PROBLEM II.

To reduce quantities of different indices to other equivalent ones, that ſhall have a common index.

RULE.

1. Divide the indices of the quantities by the given index, and the quotients will be the new indices for thoſe quantities.

2. Over the ſaid quantities, with their new indices, place the given index, and they will make the equivalent quantities required.

3. A common index may alſo be found by reducing the indices of the quantities to a common denominator, and involving each of them to the power denoted by its numerator.

EXAMPLES.

1. Reduce $15^{\frac{1}{4}}$ and $9^{\frac{1}{6}}$ to equivalent quantities having the common index $\frac{1}{2}$.

$$\frac{1}{4} \div \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{2}{4} = \frac{1}{2} = \text{1st index.}$$

$$\frac{1}{6} \div \frac{1}{2} = \frac{1}{6} \times \frac{2}{1} = \frac{2}{6} = \frac{1}{3} = \text{2d index.}$$

Therefore $(15^{\frac{1}{2}})^{\frac{1}{2}}$ and $(9^{\frac{1}{3}})^{\frac{1}{2}} =$ *quantities required.*

2. Reduce a^2 and $x^{\frac{1}{4}}$ to the ſame common index $\frac{1}{3}$.

$$\frac{2}{1}\div\frac{1}{3}=\frac{2}{1}\times\frac{3}{1}=\frac{6}{1}=\textit{1ſt index.}$$

$$\frac{1}{4}\div\frac{1}{3}=\frac{1}{4}\times\frac{3}{1}=\frac{3}{4}=\textit{2d index.}$$

Therefore $(a^6)^{\frac{1}{3}}$ and $(x^{\frac{3}{4}})^{\frac{1}{3}}$ = *quantities required.*

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Anſ. $(27)^{\frac{1}{6}}$ *and* $(4)^{\frac{1}{6}}$.

4. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

Anſ. $(a^{\frac{8}{3}})^{\frac{1}{8}}$ *and* $(b^2)^{\frac{1}{8}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to the ſame radical ſign.

Anſ. $\sqrt[nm]{a^m}$ *and* $\sqrt[nm]{b^n}$.

6. Let $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$ be reduced to a common index.

7. Let 10^2 and 100^3 be reduced to a common index.

PROBLEM III.

To reduce ſurds to their moſt ſimple terms.

RULE*.

Find the greateſt power contained in the given ſurd, and ſet its root before the remaining quantities. with the proper radical ſign between them.

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[*] When the given ſurd contains no exact power, it is already in its moſt ſimple terms.

EXAMPLES:

1. It is required to reduce $\sqrt{48}$ to its moſt ſimple terms.

$\sqrt{48} = \sqrt{(16 \times 3)} = \sqrt{16} \times \sqrt{3} = 4 \times \sqrt{3} = 4\sqrt{3}$ *the Anſwer.*

2. It is required to reduce ${}^{3}\sqrt{108}$ to its moſt ſimple terms.

${}^{3}\sqrt{108} = {}^{3}\sqrt{(27 \times 4)} = {}^{3}\sqrt{27} \times {}^{3}\sqrt{4} = 3 \times {}^{3}\sqrt{4}$ $= 3\,{}^{3}\sqrt{4}$ *the Anſwer.*

3. Reduce $\sqrt{125}$ to its moſt ſimple terms.

Anſ. $5\sqrt{5}$.

4. Reduce $\sqrt{\frac{75}{147}}$ to its moſt ſimple terms.

Anſ. $\frac{5}{7}\sqrt{3}$.

5. Reduce ${}^{3}\sqrt{81}$ to its moſt ſimple terms.

Anſ. $3\,{}^{3}\sqrt{3}$.

6. Reduce ${}^{3}\sqrt{\frac{24}{81}}$ to its moſt ſimple terms.

Anſ. $\frac{2}{3}\,{}^{3}\sqrt{3}$.

7. Reduce $\sqrt{98a^2x}$ to its moſt ſimple terms.

Anſ. $7a\sqrt{2x}$.

+ 8. Reduce $a\sqrt{(x^3 - a^2x^2)}$ to its moſt ſimple terms.

9. Reduce $(2a^3x + 3a^3x^2)^{\frac{1}{3}}$ to its moſt ſimple terms.

10. Reduce $(32a^6 - 96a^5x)^{\frac{1}{5}}$ to its moſt ſimple terms.

$2a\,\sqrt[5]{a-3x}$

PROBLEM IV.

To add ſurd quantities together.

RULE.

1. Reduce ſuch quantities as have unlike indices to other equivalent ones, having a common index.

2. Bring all fractions to a common denominator, and reduce the quantities to their ſimpleſt terms, as in the laſt problem.

3. Then, if the ſurd part be the ſame in them all, annex it to the ſum of the rational parts, with the ſign of multiplication, and it will give the total ſum required.

But if the ſurd part be not the ſame in all the quantities, they can only be added by the ſigns + and —.

EXAMPLES.

1. It is required to add $\sqrt{27}$ and $\sqrt{48}$ together.

Firſt, $\sqrt{27}=\sqrt{(9\times3)}=3\sqrt{3}$, *and* $\sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}$;

Whence, $3\sqrt{3}+4\sqrt{3}=(3+4)\sqrt{3}=7\sqrt{3}=$ *ſum required.*

2. It is required to add $\sqrt[3]{500}$, and $\sqrt[3]{108}$ together.

Firſt, $\sqrt[3]{500}=\sqrt[3]{(125\times4)}=5\sqrt[3]{4}$, *and* $\sqrt[3]{108}=\sqrt[3]{(27\times4)}=3\sqrt[3]{4}$;

Whence, $5\sqrt[3]{4}+3\sqrt[3]{4}=(5+3)\sqrt[3]{4}=8\sqrt[3]{4}=$ *ſum required.*

3. Required the ſum of $\sqrt{72}$ and $\sqrt{128}$.

Anſ. $14\sqrt{2}$.

4. Required the ſum of $\sqrt{27}$ and $\sqrt{147}$.

Anſ. $10\sqrt{3}$.

5. Required the ſum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{24}{25}}$.

Anſ. $11\sqrt{\frac{2}{75}}$.

6. Required the ſum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$.

Anſ. $5.5^{\frac{1}{3}}$.

7. Required the ſum of $\sqrt[3]{\frac{1}{32}}$ and $\sqrt[3]{\frac{16}{27}}$.

Anſ. $\frac{11}{6}.\sqrt[3]{\frac{1}{4}}$.

8. Required the ſum of $\sqrt{a^2b}$ and $\sqrt{64bx^4}$.

9. Required the ſum of $9\sqrt{243}$ and $10\sqrt{363}$.

10. It is required to find the ſum of $a^{\frac{1}{n}}$ and $a^{\frac{1}{m}}$.

11. Required the ſum of $\sqrt{27a^4x}$ and $\sqrt{3a^2x^2}$.

PROBLEM V.

To ſubtract, or find the difference, of ſurd quantities.

RULE.

Prepare the quantities as in the laſt rule, and the difference of the rational parts annexed to the common ſurd will give the difference of the ſurds required.

But if the quantities have no common ſurd, they can only be ſubtracted by means of the ſign —.

EXAMPLES.

1. It is required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

First, $\sqrt{448}=\sqrt{(64\times7)}=8\sqrt{7}$; *and* $\sqrt{112}=\sqrt{(16\times7)}=4\sqrt{7}$;

Whence $8\sqrt{7}-4\sqrt{7}=(8-4)\sqrt{7}=4\sqrt{7}=$ *difference required.*

2. It is required to find the difference of $192^{\frac{1}{3}}$ and $24^{\frac{1}{3}}$.

First, $192^{\frac{1}{3}}=(64\times3)^{\frac{1}{3}}=4.3^{\frac{1}{3}}$; *and* $24^{\frac{1}{3}}=(8\times3)^{\frac{1}{3}}=2.3^{\frac{1}{3}}$.

Whence $4.3^{\frac{1}{3}}-2.3^{\frac{1}{3}}=(4-2).3^{\frac{1}{3}}=2.3^{\frac{1}{3}}$ *difference required.*

3. Required the difference of $2\sqrt{50}$ and $\sqrt{18}$.

Anſ. $7\sqrt{2}$.

4. Required the difference of $320^{\frac{1}{3}}$ and $40^{\frac{1}{3}}$.

Anſ. $2.5^{\frac{1}{3}}$.

5. Required the difference of $\sqrt{\frac{24}{25}}$ and $\sqrt{\frac{2}{3}}$.

Anſ. $\sqrt{\frac{2}{75}}$.

6. Required the difference of $\left(\frac{16}{27}\right)^{\frac{1}{3}}$ and $\left(\frac{1}{4}\right)^{\frac{1}{3}}$.

Anſ. $\frac{1}{3}.\sqrt[3]{\frac{1}{4}}$.

7. Find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$.

Anſ. $(4a^2-2ax).\sqrt{5x}$.

8. Required the difference of $4\sqrt[3]{a^3b}$ and $3\sqrt[3]{x^2b}$.

9. It is required to find the difference of $x^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$.

PROBLEM VI.

To multiply ſurd quantities together.

RULE.

Reduce the ſurds to the ſame index, and the product of the rational quantities annexed to the product of the ſurds will give the whole product required, which may be reduced to its moſt ſimple terms by Problem 3.

EXAMPLES.

1. It is required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here $3\times2\times\sqrt{8}\times\sqrt{6}=6\sqrt{(8\times6)}=6\sqrt{48}=6\sqrt{(16\times3)}=6\times4\times\sqrt{3}=24\sqrt{3}=$ *product required.*

2. It is required to find the product of $\frac{1}{2}.\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}.\sqrt[3]{\frac{5}{6}}$.

$$\frac{1}{2}\times\frac{3}{4}\times\sqrt[3]{\frac{2}{3}}\times\sqrt[3]{\frac{5}{6}}=\frac{3}{8}\times\sqrt[3]{\frac{10}{18}}=\frac{3}{8}\times\sqrt[3]{\frac{15}{27}}=\frac{3}{8}\times\frac{1}{3}\times\sqrt[3]{15}=\frac{3}{24}\sqrt[3]{15}=\frac{1}{8}\sqrt[3]{15}=$$ *product required.*

3. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.

Anſ. $30\sqrt{10}$.

4. Required the product of $\frac{1}{2}\sqrt[3]{6}$ and $\frac{2}{3}\sqrt[3]{18}$.

Anſ. $\sqrt[3]{4}$.

5. Required the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$.

Anſ. $\frac{1}{8}\sqrt{1\frac{2}{5}}$.

6. Required the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$.

Anſ. $10.9^{\frac{1}{3}}$.

7. It is required to find the product of $a^{\frac{1}{3}}$ and $a^{\frac{2}{3}}$.

Anſ. $(a^3)^{\frac{1}{3}}$ *or* a.

8. Required the product of $(x+y)^{\frac{1}{2}}$ and $(x+y)^{\frac{1}{3}}$.

9. Required the product of $x+\sqrt{y}$ and $x-\sqrt{y}$.

10. Required the product of $(a+\sqrt{b})^{\frac{1}{2}}$ and, $(a-\sqrt{b})^{\frac{1}{2}}$.

11. It is required to find the product of $x^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$.

PROBLEM VII.

To divide one ſurd quantity by another.

RULE.

Reduce the ſurds to the ſame index, and the quotient of the rational quantities being annexed to the quotient of the ſurds will give the whole quotient required, which may be reduced to its moſt ſimple terms as before.

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

$8\div 2.\sqrt{:(108\div 6)}=4\sqrt{18}=4\sqrt{:(9\times 2)}=4\times 3\sqrt{2}=12\sqrt{2}=$ *quotient required.*

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

$8\div 4=2$, *and* $512^{\frac{1}{3}}\div 2^{\frac{1}{3}}=256^{\frac{1}{3}}=4.4^{\frac{1}{3}}$;

Therefore $2\times 4\times 4^{\frac{1}{3}}=8.4^{\frac{1}{3}}=$ *quotient required.*

3. Let $6\sqrt{100}$ be divided by $3\sqrt{2}$. *Anſ.* $10\sqrt{2}$.

4. Let $4\sqrt[3]{1000}$ be divided by $2\sqrt[3]{4}$. *Anſ.* $10\sqrt[3]{2}$.

5. Let $\frac{3}{4}\sqrt{\frac{1}{135}}$ be divided by $\frac{2}{3}\sqrt{\frac{1}{5}}$. *Anſ.* $\frac{3}{8}$ $\sqrt{\frac{1}{3}}$

6. Let $\frac{5}{7}\sqrt[3]{\frac{8}{189}}$ be divided by $\frac{2}{5}\sqrt[3]{\frac{8}{7}}$. *Anſ.* $\frac{25}{42}$.

7. Let $\frac{2}{5}\sqrt{a}$ or $\frac{2}{5}a^{\frac{1}{2}}$ be divided by $\frac{3}{4}a^{\frac{1}{3}}$. *Anſ.* $\frac{8}{15}a^{\frac{1}{6}}$.

8. Let the quantity $x^{\frac{1}{n}}$ be divided by the quantity $x^{\frac{1}{m}}$.

9. Let $x^2-xd-b+d\sqrt{b}$ be divided by $x-\sqrt{b}$.

PROBLEM VIII.

To involve ſurd quantities to any power.

RULE.

Multiply the index of the quantity by the index of the power to which it is to be raiſed, and to the reſult annex the power of the rational parts, and it will give the power required.

EXAMPLES.

1. It is required to find the ſquare of $\frac{2}{3}a^{\frac{1}{3}}$.

Firſt, $(\frac{2}{3})^2=\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$, *and* $(a^{\frac{1}{3}})^2=a^{\frac{1}{3}\times 2}=a^{\frac{2}{3}}=(a^2)^{\frac{1}{3}}$;

Whence, $(\frac{2}{3}a^{\frac{1}{3}})^2=\frac{4}{9}.(a^2)^{\frac{1}{3}}=$ *ſquare required.*

2. It is required to find the cube of $\frac{5}{7}\sqrt{7}$.

Firſt, $(\frac{5}{7})^3=\frac{5}{7}\times\frac{5}{7}\times\frac{5}{7}=\frac{125}{343}$, *and* $(7^{\frac{1}{2}})^3=7^{\frac{3}{2}}=(7^3)^{\frac{1}{2}}$;

Whence, $(\frac{5}{7}\sqrt{7})^3=\frac{125}{343}.(7^3)^{\frac{1}{2}}=\frac{125}{343}.(343)^{\frac{1}{2}}=$ *cube required.*

3. Required the ſquare of $3\sqrt[3]{3}$. *Anſ.* $9\sqrt[3]{9}$.

4. Required the cube of $2^{\frac{1}{2}}$ or $\sqrt{2}$. *Anſ.* $2\sqrt{2}$.

5. Required the 4th power of $\frac{1}{2}\sqrt{\frac{2}{3}}$. *Anſ.* $\frac{1}{36}$.

6. It is required to find the *n*th power of $a^{\frac{1}{m}}$.

7. It is required to find the ſquare of $3+\sqrt{5}$.

8. It is required to find the cube of $2x-3\sqrt{y}$.

PROBLEM IX.

To extract the roots of ſurd quantities.

RULE *.

Divide the index of the given quantity by the index of the root to be extracted, and to the reſult annex the root of the rational part, and it will give the root required.

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* The ſquare root of a binomial or reſidual ſurd $A+B$, or $A-B$, may be found thus: take $\sqrt{(A^2-B^2)}=D$; then $\sqrt{(A+B)}=\sqrt{\frac{A+D}{2}}+\sqrt{\frac{A-D}{2}}$, and $\sqrt{(A-B)}=\sqrt{\frac{A+D}{2}}-\sqrt{\frac{A-D}{2}}$. Thus the ſquare root of $8+2\sqrt{7}=1+\sqrt{7}$ and the ſquare root of $3-\sqrt{8}=\sqrt{2}-1$: but for the cube, or any higher root, no general rule can be given.

EXAMPLES.

1. It is required to find the ſquare root of $9\sqrt[3]{3}$.

Firſt, $\sqrt{9}=3$, *and* $(3^{\frac{1}{3}})^{\frac{1}{2}}=3^{\frac{1}{3}\div 2}=3^{\frac{1}{6}}$;

Whence, $(9\sqrt[3]{3})^{\frac{1}{2}}=3.3^{\frac{1}{6}}=$ *ſquare root required.*

2. It is required to find the cube root of $\frac{1}{8}\sqrt{2}$.

Firſt, $\sqrt[3]{\frac{1}{8}}=\frac{1}{2}$, *and* $(\sqrt{2})^{\frac{1}{3}}=2^{\frac{1}{2}\div 3}=2^{\frac{1}{6}}$;

Whence, $(\frac{1}{8}\sqrt{2})^{\frac{1}{3}}=\frac{1}{2}.2^{\frac{1}{6}}=$ *cube root required.*

3. Required the ſquare root of 10^3. *Anſ.* $10\sqrt{10}$.

4. Required the cube root of $\frac{8}{27}a^3$. *Anſ.* $\frac{2}{3}a$.

5. Required the 4th root of $3x^2$. *Anſ.* $3^{\frac{1}{4}}.x^{\frac{1}{2}}$.

6. It is required to find the *n*th. root of $x^{\frac{1}{m}}$.

7. Required the ſquare root of $x^2-4x\sqrt{a}+4a$.

INFINITE SERIES.

An infinite ſeries is formed from a vulgar fraction, having a compound denominator, or by extracting the root of a ſurd quantity; and is ſuch, as, being continued, would run on *ad infinitum,* in the manner of a decimal fraction.

But, by obtaining a few of the firſt terms, the law of the progreſſion will be manifeſt, ſo that the ſeries may be continued, without actually performing the whole operation.

PROBLEM I.

To reduce fractional quantities into infinite ſeries.

RULE.

Divide the numerator by the denominator, as in common diviſion; and the operation continued, as far as may be thought neceſſary, will give the ſeries required.

EXAMPLES.

1. Let $\frac{ax}{a-x}$ be propoſed to be thrown into an infinite ſeries.

$$a-x)\,ax\ldots\left(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}\ \&c.\right.$$

$$\begin{array}{l}
ax-x^2 \\ \hline
\quad x^2 \\
\quad x^2-\frac{x^3}{a} \\ \hline
\qquad \frac{x^3}{a} \\
\qquad \frac{x^3}{a}-\frac{x^4}{a^2} \\ \hline
\qquad\quad \frac{x^4}{a^2} \\
\qquad\quad \frac{x^4}{a^2}-\frac{x^5}{a^3} \\ \hline
\qquad\qquad \frac{x^5}{a^3}\ \&c.
\end{array}$$

2. Let $\frac{1}{1+x}$ be converted into an infinite ſeries.

$$\begin{array}{l}
1+x)\,1\,.....(1-x+x^2-x^3,\ \&c.\\
\quad\ \ \underline{1+x}\\
\qquad\ -x\\
\qquad\ \underline{-x-x^2}\\
\qquad\qquad\ x^2\\
\qquad\qquad\ \underline{x^2+x^3}\\
\qquad\qquad\qquad -x^3\\
\qquad\qquad\qquad \underline{-x^3-x^4}\\
\qquad\qquad\qquad\qquad\ \ x^4
\end{array}$$

3. Let $\frac{b}{a+x}$ be converted into an infinite ſeries.

Anſ. $\frac{b}{a}\times(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$ &c.

4. Let $\frac{a^2}{x+b}$ be converted into an infinite ſeries.

Anſ. $\frac{a^2}{x}\times(1-\frac{b}{x}+\frac{b^2}{x^2}-\frac{b^3}{x^3}+$ &c.)

5. Let $\frac{1+x}{1-x}$ be converted into an infinite ſeries.

Anſ. $1+2x+2x^2+2x^3+2x^4$ &c.

6. Let $\frac{a^2}{(a+x)^2}$ be converted into an infinite ſeries.

Anſ. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}$ &c.

7. Let $\frac{1}{2}$ or its equal $\frac{1}{1+1}$ be converted into an infinite ſeries.

PROBLEM II.

To reduce a compound ſurd into an infinite ſeries.

RULE *.

Extract the root as in common arithmetic, and the operation, continued as far as may be thought neceſſary, will give the ſeries required.

EXAMPLES.

1. Extract the ſquare root of a^2+x^2 in an infinite ſeries.

$$\begin{array}{rl}
 & a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\ \&c.\right. \\
 & a^2 \\
\hline
2a+\frac{x^2}{2a}\Big) & x^2 \\
 & x^2+\frac{x^4}{4a^2} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big) & -\frac{x^4}{4a^2} \\
 & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\ \&c.\Big) & \frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
 & \frac{x^6}{8a^4}+\frac{x^8}{16a^6}\ \&c. \\
\hline
 & -\frac{5x^8}{64a^6}\ \&c.
\end{array}$$

---

[*] This rule is chiefly of uſe in extracting the ſquare root, the operation being too tedious for the higher powers.

2. Let $\sqrt{2}$ or $\sqrt{1+1}$ be thrown into an infinite ſeries.

Anſ. $1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$ *&c.*

3. Let $\sqrt{a^2-x^2}$ be thrown into an infinite ſeries.

Anſ. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}$ *&c.*

4. Let $\sqrt[3]{1-x^3}$ be thrown into an infinite ſeries.

Anſ. $1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}$ *&c.*

5. Let $\sqrt{a^2+b}$ be converted into an infinite ſeries.

PROBLEM III.

To reduce a binomial ſurd into an infinite ſeries; or to extract any root of a binomial.

RULE *.

Subſtitute the particular letters of the binomial, with their proper ſigns, in the following general form, and it will give the root required; obſerving that P is the firſt term, Q the ſecond term divided by the firſt, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c. the foregoing terms with their proper ſigns.

$$\overline{P+PQ}^{\frac{m}{n}}=P^{\frac{m}{n}}(A)+\frac{m}{n}AQ(B)+\frac{m-n}{2n}BQ(C)+\frac{m-2n}{3n}CQ(D)+\frac{m-3n}{4n}DQ(E)\ \&c.$$

---

* When the index $\frac{m}{n}$ becomes an integer, the rule is the ſame as that before given for the raiſing of powers, and *vice verſa*.

EXAMPLES.

1. To extract the ſquare root of r^2-x^2, in an infinite ſeries.

Here $\text{P}=r^2$; $\text{Q}=-\frac{x^2}{r^2}$, *and* $\frac{m}{n}=\frac{1}{2}$; *therefore,* $(r^2-x^2)^{\frac{1}{2}}=r+(\frac{1}{2}\times\text{A}\times-\frac{x^2}{r^2})+(-\frac{1}{4}\times\text{B}\times-\frac{x^2}{r^2})+(-\frac{3}{6}\times\text{C}\times-\frac{x^2}{r^2})+(-\frac{5}{8}\times\text{D}\times-\frac{x^2}{r^2})$ *&c.* $=r+(-\frac{x^2}{2r^2}.\text{A})+(\frac{x^2}{4r^2}.\text{B})+(\frac{3x^2}{6r^2}.\text{C})+(\frac{5x^2}{8r^2}.\text{D})$ *&c:* $=r-\frac{x^2}{2r}\text{A}+\frac{x^2}{4r^2}\text{B}+\frac{3x^2}{6r^2}\text{C}+\frac{5x^2}{8r^2}\text{D}$ *&c. which, by reſtoring the values of* A, B, C, D, *&c. becomes* $r-\frac{x^2}{2r}-\frac{x^4}{8r^3}-\frac{x^6}{16r^5}-\frac{5x^8}{128r^7}$, *&c.* = *ſeries required.*

2. To find the value of $\frac{1}{(a+b)^2}$ or its equal $(a+b)^{-2}$ in an infinite ſeries.

Here $\text{P}=a$, $\text{Q}=\frac{b}{a}$, and $\frac{m}{n}=\frac{-2}{1}$; *therefore* $(a+b)^{-2}=a^{-2}+(-\frac{2}{1}.\text{A}.\frac{b}{a})+(-\frac{3}{2}.\text{B}.\frac{b}{a})+(-\frac{4}{3}.\text{C}.\frac{b}{a})+(-\frac{5}{4}.\text{D}.\frac{b}{a})$ *&c.* $=\frac{1}{a^2}+(-\frac{2b}{a}.\text{A})+(-\frac{3b}{2a}.\text{B})+(-\frac{4b}{3a}.\text{C})+(-\frac{5b}{4a}.\text{D})$ *&c.* $=\frac{1}{a^2}-\frac{2b}{a}\text{A}+\frac{3b}{2a}\text{B}-\frac{4b}{3a}\text{C}-\frac{5b}{4a}\text{D}$, *&c. which, by reſtoring the values of* A, B, C, D, *becomes* $\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b}{a^4}-\frac{4b}{a^5}+\frac{5b}{a^6}$, *&c.* = *ſeries required.*

3. To find the value of $\frac{r^2}{r+x}$, in an infinite ſeries.

Anſ. $r-x+\frac{x^2}{r}-\frac{x^3}{r^2}+\frac{x^4}{r^3}$, &c.

4. To find the value of $\frac{1}{(a^2-x^2)^{\frac{1}{2}}}$ in an infinite ſeries.

Anſ. $\frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}+\frac{15x^6}{48a^7}$, &c.

5. To find the value of $\frac{a^2}{(a+x)^2}$ in an infinite ſeries.

Anſ. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\frac{5x^4}{a^4}$, &c.

6. To find the value of $(a^2+b)^{\frac{1}{2}}$ in an infinite ſeries. *Anſ.* $a+\frac{b}{2a}-\frac{b^2}{8a^3}+\frac{b^3}{16a^5}-\frac{5b^4}{128a^7}$, &c.

7. Find the value of $(a^2-x^2)^{\frac{1}{5}}$ in an infinite ſeries.

Anſ. $a^{\frac{2}{5}}\times\left(1-\frac{x^2}{5a^2}-\frac{2x^4}{25a^4}-\frac{6x^6}{125a^6}\right.$, &c.

8. To find the value of $(a^3-b)^{\frac{1}{3}}$ in an infinite ſeries.

Anſ. $a-\frac{b}{3a^2}-\frac{b^2}{9a^5}-\frac{5b^3}{81a^8}-\frac{10b^4}{243a^{11}}$, &c.

9. Required the ſquare root of $\frac{a^2+x^2}{a^2-x^2}$ in an infinite ſeries.

Anſ. $1+\frac{x^2}{a^2}+\frac{x^4}{2a^4}+\frac{x^6}{2a^6}$, &c.

10. Required the cube root of $\frac{a^2}{(a^2+x^2)^2}$ in an infinite ſeries. *Anſ.* $\frac{1}{a^{\frac{\text{[illegible]}}{3}}}\times\left\{1-\frac{2x^2}{3a^2}+\frac{5x^4}{9a^4}-\frac{40x^6}{81a^6}\right.$, &c.

11. Required the value of $\frac{ax}{a^2-ax+x^2}$, in an infinite ſeries.

Anſ. $\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^4}{a^4}-\frac{x^5}{a^5}$, &c.

ARITHMETICAL PROPORTION.

Arithmetical proportion is the relation which two quantities, of the ſame kind, bear to each other, with reſpect to their difference.

Four quantities are ſaid to be in *arithmetical proportion*, when the difference between the firſt and ſecond is equal to the difference between the third and fourth.

Thus, 3, 7, 12, 16, *and* a, $a+b$, c, $c+b$, *are arithmetically proportional.*

Arithmetical progreſſion is when a ſeries of quantities either increaſe or decreaſe by the ſame common difference.

Thus, 1, 3, 5, 7, 9, 11, &c. *and* a, $a+b$, $a+2b$, $a+3b$, $a+4b$, $a+5b$, &c. *are ſeries in arithmetical progreſſion, whoſe common differences are* 2 *and* b.

The moſt uſeful part of arithmetical proportion is contained in the following theorems:

I. If four quantities be in arithmetical proportion, the ſum of the two means will be equal to the ſum of the two extremes.

Thus, if 2, 5, 7, 10, *and* a, b, c, d, *are in arithmetical proportion, then will* $2+10=5+7$, *and* $a+d=b+c$.

II. In any continued arithmetical progreſſion, the ſum of the two extremes, and that of any two terms which are equally diſtant from them, are equal to each other.

Thus, in the ſeries, 2, 4, 6, 8, 10, 12, *&c.* $2+12=4+10=6+8$.

III. The laſt term of any arithmetical ſeries, is equal to the ſum, or difference, of the firſt term, and the product of the common difference, by the number of terms leſs one; according as the ſeries is increaſing or decreaſing.

Thus, the 20th term of 2, 4, 6, 8, 10, 12, *&c. is* $=2+(20-1)\times2=2+19\times2=2+38=40$,

And the nth term of a, $a-x$, $a-2x$, $a-3x$, $a-4x$, *&c. is* $=a-(n-1)\times x=a-(n-1)x$.

IV. The ſum of any ſeries of quantities in arithmetical progreſſion is equal to the ſum of the two extremes multiplied by half the number of terms.

Thus, the ſum of 1, 2, 3, 4, 5, 6, *&c. continued to the 20th term, is* $=\frac{(1+20)\times20}{2}=\frac{21\times20}{2}=21\times10=210$.

And the ſum of n terms of a, $a+x$, $a+2x$, $a+3x$ *to* $a+mx$ *is* $=(a+a+mx).\frac{n}{2}=(a+\frac{1}{2}mx).n=(a+\frac{n-1}{2}x)n$.

EXAMPLES.

1. The firſt term of an increaſing arithmetical ſeries is 3, the common difference 2, and the number of terms 20; required the ſum of the ſeries.

Firſt, $3+2\times(20-1)=3+2\times(19)=3+38=41=$ *laſt term.*

And, $(3+41)\times\frac{20}{2}=44\times\frac{20}{2}=44\times10=440=$ *ſum required.*

2. The firſt term of a decreaſing arithmetical ſeries is 100, the common difference 3, and the number of terms 34; required the ſum of the ſeries.

Firſt, $100-3.(34-1)=100-3.(33)=100-99$ $=1=$ *laſt term.*

And, $(100+1)\times\frac{34}{2}=101\times\frac{34}{2}=101\times17=1717$ $=$ *ſum required.*

3. Required the ſum of the natural numbers 1, 2, 3, 4, 5, 6, &c. continued to 1000 terms.
Anſ. 500500.

4. * Required the ſum of the odd numbers 1, 3, 5, 7, 9, &c. continued to 101 terms. *Anſ.* 10201.

5. How many ſtrokes do the clocks of Venice, which go on to 24 o'clock, ſtrike in the compaſs of a day? *Anſ.* 300.

6. The firſt term of a decreaſing arithmetical ſeries is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the ſum of the ſeries. *Anſ.* 175.

7. One hundred ſtones being placed on the ground, in a ſtraight line, at the diſtance of a yard from each other, how far will a perſon travel who ſhall bring them one by one to a baſket, which is placed one yard from the firſt ſtone.
Anſ. 5 *miles and* 1300 *yards.*

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* The ſum of any number of terms (n) of the arithmetical ſeries of odd numbers 1, 3, 5, 7, 9, &c. is equal to the ſquare (n^2) of that number.

That is, if 1, 3, 7, 9, &c. be the numbers,

Then will 1, 2^2, 3^2, 4^2, 5^2, &c. be the ſums of 1, 2, 3, &c. terms.

For, $0+1$, or the ſum of 1 term $=1^2$, or 1
$1+3$, or the ſum of 2 terms $=2^2$, or 4
$4+5$, or the ſum of 3 terms $=3^2$, or 9
$9+7$, or the ſum of 4 terms $=4^2$, or 16, &c.

Whence it is plain, that, let n be any number whatever, the ſum of n terms will be n^2.

GEOMETRICAL PROPORTION.

Geometrical proportion is that relation of two quantities of the ſame kind, which ariſes from conſidering what part the one is of the other, or how often it is contained in it.

When four quantities are compared together, the firſt and third are called the *antecedents*, and the ſecond and fourth the *conſequents*.

Ratio is the quotient which ariſes from dividing the antecedent by the conſequent, or the conſequent by the antecedent.

Four quantities are ſaid to be proportional, when the firſt is the ſame part or multiple of the ſecond, as the third is of the fourth.

Thus, 2, 8, 3, 12, *and* a, ar, b, br, *are geometrical proportionals*.

Direct proportion is when the ſame relation ſubſiſts between the firſt term and the ſecond, as between the third and the fourth.

Thus, 3, 6, 5, 10, *and* x, ax, y, ay, *are in direct proportion*.

Reciprocal, or inverſe proportion, is when one quantity increaſes in the ſame proportion as another diminiſhes. or indirect

Thus, 2, 6, 9, 3, *and* a, ar, br, b, *are in inverſe proportion*.

A ſeries of quantities are ſaid to be in geometrical progreſſion, when the firſt has the ſame ratio to the ſecond as the ſecond to the third, the third to the fourth, &c.

Thus, 2, 4, 8, 16, 32, 64, *&c. and* a, ar, ar^2, ar^3, ar^4, ar^5, *&c. are ſeries in geometrical progreſſion*.

The moſt uſeful part of geometrical proportion is contained in the following theorems.

I. If four quantities be in geometrical proportion the product of the two means will be equal to that of the two extremes.

Thus, if 2, 4, 6, 12, *and* a, ar, b, br, *be geometrically proportional, then will* $2\times12=4\times6$, *and* $a\times br=b\times ar$.

II. If four quantities be in geometrical proportion, the rectangle of the means divided by either of the extremes will give the other extreme.

Thus, if 3, 9, 5, 15, *and* x, ax, y, ay, *are geometrical proportionals, then will* $\frac{9\times5}{3}=15$, *and* $\frac{ax\times y}{ay}=x$.

III. In any continued geometrical progreſſion, the product of the two extremes, and that of any other two terms, equally diſtant from them, will be equal to each other.

Thus, in the ſeries 1, 3, 9, 27, 81, 243, *&c.* $1\times243=3\times81=9\times27$.

IV. In any continued geometrical ſeries, the laſt term is equal to the firſt multiplied by ſuch a power of the ratio as is denoted by the number of terms leſs one.

Thus, in the ſeries 2, 6, 18, 54, 162, *&c.* $2\times3^4=162$.

V. The ſum of any ſeries in geometrical progreſſion is found by multiplying the laſt term by the ratio, and dividing the difference of this product and the firſt term by the ratio leſs one.

Thus, the ſum of 2, 4, 8 16, 32, 64, 128, 256, 512, *is* $\frac{512\times2-2}{2-1}=1024-2=1022$.

And the ſum of n *terms of* a, ar, ar^2, ar^3, ar^4, *&c. to* ar^{n-1}, *is* $\frac{ar^{n-1}\times r-a}{r-1}=\frac{ar^n-a}{r-1}$.

VI. If four quantities, a, b, c, d, or 2, 6, 5, 15, be proportional, then will any of the following forms of thoſe quantities be alſo proportional.

1. *directly* $a : b :: c : d$ or $2 : 6 :: 5 : 15$.
2. *inverſely* $b : a :: d : c$ or $6 : 2 :: 15 : 5$.
3. *alternately* $a : c :: b : d$ or $2 : 5 :: 6 : 15$.
4. *compoundedly* $a : a+b :: c : c+d$ or $2 : 8 :: 5 : 20$.
5. *dividedly* $a : b-a :: c : d-c$ or $2 : 4 :: 5 : 10$.
6. *mixed* $b+a : b-a :: d+c : d-c$ or $8 : 4 :: 20 : 10$.
7. *by multiplication* $ra : rb :: c : d$ or $2\times 3 : 6\times 3 : 5 : 15$.
8. *by diviſion* $a\div r : b\div r :: c : d$ or $1 : 3 :: 5 : 15$.
9. *When* $a \backsim b : c \backsim d :: a : d$ *the numbers* a, b, c, d, *are in harmonical proportion.*

EXAMPLES.

1. The firſt term of a geometrical ſeries is 1, the ratio 2, and the number of terms 10; what is the ſum of the ſeries?

Firſt, $1\times 2^9 = 1\times 512 = 512 =$ *laſt term.*

And, $\frac{512\times 2-1}{2-1} = \frac{1024-1}{1} = 1023 =$ *ſum required.*

2. The firſt term of a geometric ſeries is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5; required the ſum of the ſeries.

Firſt, $\frac{1}{2}\times(\frac{1}{3})^4 = \frac{1}{2}\times\frac{1}{81} = \frac{1}{162} =$ *laſt term.*

And, $(\frac{1}{2} - \frac{1}{162}\times\frac{1}{3}) \div 1 - \frac{1}{3} = (\frac{1}{2} - \frac{1}{486}) \div \frac{2}{3} = \frac{21}{243} \times\frac{3}{2} = \frac{121}{81\times 2} = \frac{121}{162} =$ *ſum required.*

3. Required the ſum of 1, 3, 9, 27, 81, &c. continued to 12 terms. *Anſ.* 265720.

4. Required the ſum of $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, &c. continued to 12 terms.

5. Required the ſum of 1, 2, 4, 8, 16, 32, &c. continued to 100 terms.

SIMPLE EQUATIONS.

An equation is, when two equal quantities, differently expreſſed, are compared together by means of the ſign $=$ placed between them.

Thus $12-5=7$ is an equation, expreſſing the equality of the quantities $12-5$ and 7.

A ſimple equation is that which contains only one unknown quantity, without including its power.

Thus $x-a+b=c$ is a ſimple equation, containing only the unknown quantity x.

Reduction of equations is the method of finding the value of the unknown quantity; which is ſhewn in the following rules.

RULE I.

Any quantity may be tranſpoſed from one ſide of the equation to the other by changing its ſign.

Thus, if $x+3=7$, *then will* $x=7-3=4$.
And, if $x-4+6=8$, *then will* $x=8+4-6=6$.
Alſo, if $x-a+b=c-d$, *then will* $x=c-d+a-b$.
And, in like manner, if $4x-8=3x+20$, *then will* $4x-3x=20+8$, *or* $x=28$.

RULE II.

If the unknown term be multiplied by any quantity, it may be taken away by dividing all the other terms of the equation by it.

Thus, if $ax=ab-a$, *then will* $x=b-1$.

And, if $2x+4=16$, *then will* $x+2=8$, *and* $x=8-2=6$.

In like manner, if $ax+2ba=3c^2$, *then will* $x+2b=\frac{3c^2}{a}$, *and* $x=\frac{3c^2}{a}-2b$.

RULE III.

If the unknown term be divided by any quantity, it may be taken away by multiplying all the other terms of the equation by it.

Thus, if $\frac{x}{2}=5+3$, *then will* $x=10+6=16$.

And, if $\frac{x}{a}=b+c-d$, *then will* $x=ab+ac-ad$.

In like manner, if $\frac{2x}{3}-2=6+4$, *then will* $2x-6=18+12$, *and* $2x=18+12+6=36$, *or* $x=\frac{36}{2}=18$.

RULE IV.

The unknown quantity in any equation may be made free from ſurds, by tranſpoſing the reſt of the terms by Rule 1, and then involving each ſide to ſuch a power as is denoted by the index of the ſaid ſurd.

Thus, if $\sqrt{x}-2=6$, *then will* $\sqrt{x}=6+2=8$, *and* $x=8^2=64$.

And, if $\sqrt{4x+16}=12$, *then will* $4x+16=144$, *or* $4x=144-16=128$; *and if both ſides of the equation be divided by* 4, x *will be* $=32$.

In like manner, if $\sqrt[3]{2x+3}+4=8$, *then will* $\sqrt[3]{2x+3}=8-4=4$, *and* $2x+3=4^3=64$, *and* $2x=64-3=61$, *or* $x=\frac{61}{2}=30\frac{1}{2}$.

RULE V.

If that ſide of the equation which contains the unknown quantity be a complete power, it may be reduced by extracting the root of the ſaid power from both ſides of the equation.

Thus, if $x^2+6x+9=25$, *then will* $x+3=\sqrt{25}=5$, *or* $x=5-3=2$.

And, if $3x^2-9=21+3$, *then will* $3x^2=21+3+9=33$, *and* $x^2=\frac{33}{3}=11$, *or* $x=\sqrt{11}$.

In like manner, if $\frac{2x^2}{3}+10=20$, *then will* $2x^2+30=60$, *and* $x^2+15=30$, *or* $x^2=30-15=15$, *or* $x=\sqrt{15}$.

RULE VI.

Any analogy or proportion may be converted into an equation, by making the product of the two mean terms equal of that of the two extremes.

Thus, if $3x : 16 :: 5 : 10$, *then will* $3x\times10=16\times5$, *and* $30x=80$, *or* $x=\frac{80}{30}=2\frac{2}{3}$.

And, if $\frac{2x}{3} : a :: b : c$, *then will* $\frac{2cx}{3}=ab$, *and* $2cx=3ab$, *or* $x=\frac{3ab}{2c}$.

In like manner, if $12-x:\frac{x}{2}::4:1$, *then will* $12-x=\frac{4x}{2}=2x$, *and* $2x+x=12$, *or* $x=\frac{12}{3}=4$.

RULE VII.

If any quantity be found on both ſides of the equation with the ſame ſign, it may be taken away from each; and if every term in an equation be multiplied or divided by the ſame quantity, it may be ſtruck out of them all.

Thus, if $4x+a=b+a$, *then will* $4x=b$, *and* $x=\frac{b}{4}$.

And, if $3ax+5ab=8ac$, *then will* $3x+5b=8c$, *and* $x=\frac{8c-5b}{3}$.

In like manner, if $\frac{2x}{3}-\frac{8}{3}=\frac{16}{3}-\frac{8}{3}$, *then will* $2x=16$, *and* $x=8$.

MISCELLANEOUS EXAMPLES.

1. Given $5x-15=2x+6$ to find the value of x.

   Firſt $5x-2x=6+15$
   then $3x=21$
   and $x=\frac{21}{3}=7$.

2. Given $40-6x-16=120-14x$ to find x.

   Firſt $14x-6x=120-40+16$
   then $8x=96$
   and therefore $x=\frac{96}{8}=12$.

3. Let $5ax-3b=2dx+c$ be given, to find x.
Firſt, $5ax-2dx=c+3b$
or $(5a-2d)\times x=c+3b$
and therefore $x=\frac{c+3b}{5a-2d}$.

4. Let $3x^2-10x=8x+x^2$ be given to find x.
Firſt, $3x-10=8+x$
and then $3x-x=8+10$
therefore $2x=18$, *and* $x=\frac{18}{2}=9$.

5. Given $6ax^3-12abx^2=3ax^3+6ax^2$, to find x.
Firſt, dividing the whole by $3ax^2$, *we ſhall have* $2x-4b=x+2$
and then $2x-x=2+4b$
whence $x=2+4b$.

6. Let $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10$, be given to find x.
Firſt, $x-\frac{2x}{3}+\frac{2x}{4}=20$
and then $3x-2x+\frac{6x}{4}=60$
and $12x-8x+6x=240$
therefore $10x=240$
and $x=\frac{240}{10}=24$.

7. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$ to find x.
Firſt, $x-3+\frac{2x}{3}=40-x-19$.
and then $3x-9+2x=120-3x-57$
and therefore $3x+2x+3x=120-57+9$
that is $8x=72$, *or* $x=\frac{72}{8}=9$.

8. Let $\sqrt{\frac{2}{3}x}+5=7$, be given to find x.

Firſt, $\sqrt{\frac{2}{3}x}=7-5=2$
and then $\frac{2}{3}x=2^2=4$
and $2x=12$, *or* $x=\frac{12}{2}=6$.

9. Let $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$ be given to find x.

Firſt, $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$
and then $x\sqrt{a^2+x^2}=a^2-x^2$
and $x^2\times(a^2+x^2)=(a^2-x^2)^2=a^4-2a^2x^2+x^4$
or $a^2x^2+x^4=a^4-2a^2x^2+x^4$
whence $a^2x^2+2a^2x^2=a^4$, *or* $3a^2x^2=a^4$
and conſequently $x^2=\frac{a^4}{3a^2}$, *and* $x=\sqrt{\frac{a^4}{3a^2}}=a\sqrt{\frac{1}{3}}$.

EXAMPLES FOR PRACTICE.

1. Given $3y-2+24=31$ to find y. *Anſ.* $y=3$.

2. Given $x+18=3x-5$ to find x. *Anſ.* $x=11\frac{1}{2}$.

3. Given $6-2x+10=20-3x-2$ to find x.
Anſ. $x=2$.

4. Given $x+\frac{1}{2}x+\frac{1}{3}x=11$ to find x. *Anſ.* $x=6$.

5. Given $2x-\frac{1}{2}x+1=5x-2$ to find x. *Anſ.* $x=\frac{6}{7}$.

6. Given $3ax+\frac{a}{2}-3=bx-a$ to find x.
Anſ. $x=\frac{6-3a}{6a-2b}$.

7. Given $\frac{1}{2}x+\frac{1}{3}x-\frac{1}{4}x=\frac{1}{2}$ to find x. *Anſ.* $x=\frac{6}{7}$.

8. Given $\sqrt{:}(12+x)=2+\sqrt{x}$ to find x. *Anſ.* $x=4$.

9. Given $x+a=\frac{x^2}{a+x}$ to determine x. *Anſ.* $x=-\frac{1}{2}a$

10. Given $\sqrt{:}(a^2+x^2)=(b^4+x^4)^{\frac{1}{4}}$ to find x.

Anſ. $x=\sqrt{\frac{b^4-a^4}{2a^2}}$.

11. Given $\sqrt{x}+\sqrt{:}(a+x)=\frac{2a}{\sqrt{:}(a+x)}$ to find x.

Anſ. $x=\frac{1}{3}a$.

12. Given $x+\sqrt{:}(a^2+x^2)=\frac{2a^2}{\sqrt{:}(a^2+x^2)}$ to find x.

Anſ. $\frac{1}{3}a\sqrt{3}$. *

13. Given $x+a=\sqrt{:}(a^2+x\sqrt{(b^2+x^2)})$ to find x.

Anſ. $x=\frac{b^2}{4a}-a$.

PROBLEM I.

To exterminate two unknown quantities, or to reduce the two ſimple equations containing them, to a ſingle one.

RULE I.

1. Obſerve which of the unknown quantities is the leaſt involved, and find its value in each of the equations, by the methods already explained.

2. Let the two values thus found be made equal to each other, and there will ariſe a new equation with only one unknown quantity in it, whoſe value may be found as before.

* $x = a\sqrt{\frac{1}{3}} = a\sqrt{\frac{3}{9}} = \frac{a}{3}\sqrt{3}$

EXAMPLES.

1. Given $\left\{\begin{array}{l}2x+3y=23\\5x-2y=10\end{array}\right\}$ to find x and y.

From the firſt equation $x=\frac{23-3y}{2}$,

and from the ſecond $x=\frac{10+2y}{5}$,

conſequently $\frac{23-3y}{2}=\frac{10+2y}{5}$,

or $115-15y=20+4y$,

or $19y=115-20=95$,

and $y=\frac{95}{19}=5$,

whence $x=\frac{23-15}{2}=4$.

2. Given $\left\{\begin{array}{l}x+y=a\\x-y=b\end{array}\right\}$ to find x and y.

From the firſt equation $x=a-y$,
and from the ſecond $x=b+y$,
Therefore $a-y=b+y$, *or* $2y=a-b$,
conſequently $y=\frac{a-b}{2}$, *and* $x=a-y$

$$=a-\frac{a-b}{2}=\frac{a+b}{2}.$$

3. Given $\left\{\begin{array}{l}\frac{1}{2}x+\frac{1}{3}y=7\\\frac{1}{3}x+\frac{1}{2}y=8\end{array}\right\}$ to find x and y.

From the firſt equation $x=14-\frac{2y}{3}$,

and from the ſecond $x=24-\frac{3y}{2}$, *Therefore* $14-\frac{2y}{3}=24-\frac{3y}{2}$,

and $42-2y=72-\frac{9y}{2}$,

or $84-4y=144-9y$;

whence $5y=144-84=60$,

and $y=\frac{60}{5}=12$,

and $x=14-\frac{2y}{3}=14-\frac{24}{3}=6$.

4. Given $4x+y=34$, and $4y+x=16$, to find x and y.
Anſ. $x=8$, *and* $y=2$.

5. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$, to find x and y.
Anſ. $x=\frac{1}{2}$, *and* $y=\frac{1}{3}$.

6. Given $x+y=s$, and $x^2-y^2=d$, to find x *and* y.
Anſ. $x=\frac{s^2+d}{2s}$, *and* $y=\frac{s^2-d}{2s}$.

7. Given $x-y=d$, and $x:y::n:m$, to find x and y.

RULE II.

1. Conſider which of the unknown quantities you would firſt exterminate, and let its value be found in that equation where it is the leaſt involved.

2. Subſtitute the value, thus found, for its equal in the other equation, and there will ariſe a new equation, with only one unknown quantity, whoſe value may be found as before.

EXAMPLES.

1. Given $\left\{\begin{array}{r} x+2y=17 \\ 3x-y=2 \end{array}\right\}$ to find x and y.

From the firſt equation $x=17-2y$,

And this value ſubſtituted for x in the ſecond, gives

$(17-2y)\times 3-y=2$,

or $51-6y-y=2$, *or* $51-7y=2$;

that is $7y=51-2=49$;

whence $y=\frac{49}{7}=7$, *and* $x=17-2y=17-14=3$.

2. Given $\left\{\begin{array}{r} x+y=13 \\ x-y=3 \end{array}\right\}$ to find x and y.

From the firſt equation $x=13-y$,

And this value being ſubſtituted for x in the 2d,

gives $13-y-y=3$, *or* $13-2y=3$,

that is $2y=13-3=10$,

whence $y=\frac{10}{2}=5$, *and* $x=13-y=13-5=8$.

3. Given $\left\{\begin{array}{l} a:b::x:y \\ x^2+y^2=c \end{array}\right\}$ to find x and y.

The firſt analogy turned into an equation

is $bx=ay$, *or* $x=\frac{ay}{b}$,

And this value of x being ſubſtituted in the 2d,

gives $(\frac{ay}{b})^2+y^2=c$, *or* $\frac{a^2y^2}{b^2}+y^2=c$,

or $a^2y^2+b^2y^2=cb^2$, *or* $y^2=\frac{cb^2}{a^2+b^2}$,

whence $y=(\frac{cb^2}{a^2+b^2})^{\frac{1}{2}}$, *and* $x=(\frac{ca^2}{a^2+b^2})^{\frac{1}{2}}$.

4. Given $2x+3y=16$, and $3x-2y=11$ to find x and y. *Anſ.* $x=5$ *and* $y=2$.

5. Given $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$, to find x and y. *Anſ.* $x=7$ *and* $y=14$.

6. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, to find x and y. *Anſ.* $x=60$ *and* $y=40$.

7. Given $a:b::x:y$, and $x^3-y^3=d$, to find x and y. *Anſ.* $\left(\frac{da^3}{a^3-b^3}\right)^{\frac{1}{3}}=x$, *and* $\left(\frac{db^3}{a^3-b^3}\right)^{\frac{1}{3}}=y$.

RULE III.

1. Let the given equations be multiplied or divided by ſuch numbers or quantities as will make the term which contains one of the unknown quantities the ſame in both equations.

2. Then by adding or ſubtracting the equations, according as is required, there will ariſe a new equation with only one unknown quantity, as before.

EXAMPLES.

1. Given $\left\{\begin{array}{r}3x+5y=40\\ x+2y=14\end{array}\right\}$ to find x and y.

Firſt multiply the 2d equation by 3, *and it will give* $3x+6y=42$,

Then, from this laſt equation ſubtract the firſt, and it will give $6y-5y=42-40$, *or* $y=2$, *and therefore* $x=14-2y=14-4=10$.

2. Given $\left\{\begin{array}{l} 5x-3y=9 \\ 2x+5y=16 \end{array}\right\}$ to find x and y.

Let the 1ſt equation be multiplied by 2, and the 2d by 5, and we ſhall have $10x-6y=18$
$10x+25y=80$

and if the former of theſe be ſubtracted from the latter, it will give $31y=62$, *or* $y=\frac{62}{31}=2$,

and conſequently $x=\frac{9+3y}{5}$, *by the firſt equation,*

$$\text{or } x=\frac{9+6}{5}=\frac{15}{5}=3.$$

Another method.

Multiply the firſt equation by 5, and the ſecond by 3, and we ſhall have $25x-15y=45$
$6x+15y=48$ = $2x+5y=16$

Now, let theſe two equations be added together, and it will give $31x=93$, *or* $x=\frac{93}{31}=3$,

and conſequently $y=\frac{16-2x}{5}$, *by the ſecond equation,*

$$\text{or } y=\frac{16-6}{5}=\frac{10}{5}=2 \text{ as before.}$$

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x+2}{3}+8y=31$, and $\frac{y+5}{4}+10x=192$, to find x and y. *Anſ.* $x=19$ *and* $y=3$.

2. Given $\frac{2x-y}{2}+14=18$, and $\frac{2y+x}{3}+16=19$, to find x and y. *Anſ.* $x=5$ *and* $y=2$.

3. Given $\frac{2x+3y}{6}+\frac{x}{3}=8$, and $\frac{7y-3x}{2}-y=11$, to find x and y. *Anſ.* $x=6$ *and* $y=8$.

4. Given $ax+by=c$, and $dx+ey=f$, to find x and y.

Anſ. $x=\frac{ce-bf}{ae-db}$ *and* $y=\frac{af-dc}{ae-db}$.

PROBLEM II.

To exterminate three unknown quantities, or to reduce the three ſimple equations, containing them, to a ſingle one.

RULE.

1. Let x, y, and z, be the three unknown quantities to be exterminated.

2. Find the value of x from each of the three given equations.

3. Compare the firſt value of x with the ſecond, and an equation will ariſe involving only y and z.

4. In like manner, compare the firſt value of x with the third, and another equation will ariſe involving only y and z.

5. Find the values of y and z from theſe two equations, according to the former rules, and x, y, and z, will be exterminated as required.

Note. Any number of unknown quantities may be exterminated in nearly the ſame manner; but there are often much ſhorter methods for performing the operation, which will be beſt learnt from practice.

EXAMPLES.

1. Given $\left\{\begin{array}{r} x+y+z=29\\ x+2y+3z=62\\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10\end{array}\right\}$ to find x, y, and z.

From the firſt equation $x=29-y-z$.

From the ſecond $x=62-2y-3z$.

From the third $x=20-\frac{2y}{3}-\frac{z}{2}$.

whence $29-y-z=62-2y-3z$,

and $29-y-z=20-\frac{2y}{3}-\frac{z}{2}$,

but, from the firſt of theſe equations, $y=62-29-2z=33-2z$,

and from the ſecond $y=27-\frac{3z}{2}$,

therefore $33-2z=27-\frac{3z}{2}$, *or* $z=12$,

and $y=62-29-2z=62-29-24=9$,
and $x=29-y-z=29-12-9=8$.

2. Given $\left\{\begin{array}{r} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62\\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47\\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38\end{array}\right\}$ to find x, y, and z.

Firſt, the given equations, cleared of fractions, become

$$\begin{array}{r} 12x+8y+6z=1488\\ 20x+15y+12z=2820\\ 30x+24y+20z=4560\end{array}$$

Then, if the ſecond of theſe equations be ſubtracted from double the firſt, and three times the third from five times the ſecond, we ſhall have

$$\begin{array}{r} 4x+y=156\\ 10x+3y=420\end{array}$$

And again, if the ſecond of theſe be ſubtracted from three times the firſt, it will give

$$12x-10x=468-420,\ \textit{or}\ x=\frac{48}{2}=24$$

Therefore $y=156-4x=60$, *and* $z=\frac{1488-8y-12x}{6}=120$.

3. Given $x+y+z=31$, $2x+3y+4z=76$, and $3x+4y+5z=107$, to find x, y, and z.

Anſ. $x=20$, $y=8$, *and* $z=3$.

4. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find x, y, and z.

5. Given $\left\{\begin{array}{l} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{array}\right\}$ to find x, y, and z.

A COLLECTION OF QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. To find two numbers, ſuch that their ſum ſhall be 40, and their difference 16.

Let x *denote the leaſt of the two numbers required,*
then will $x+16=$ *to the greater,*
and $x+x+16=40$ *by the queſtion,*
that is $2x=40-16=24$,
or $x=\frac{24}{2}=12=$ *leaſt number,*
and $x+16=12+16=28=$ *greater number required.*

2. What number is that whoſe $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let $x =$ *number required,*

then will its $\frac{1}{3}$ *part be* $\frac{x}{3}$, *and its* $\frac{1}{4}$ *part* $\frac{x}{4}$;

and therefore $\frac{x}{3} - \frac{x}{4} = 16$ *by the queſtion,*

that is $x - \frac{3x}{4} = 48$, *or* $4x - 3x = 192$;

whence $x = 192$ *the number required.*

3. Divide 1000l. between A, B, and C, ſo that A ſhall have 72l. more than B, and C 100l. more than A.

Let $x =$ B's *ſhare of the given ſum,*
then will $x + 72 =$ A's *ſhare,*
and $x + 172 =$ C's *ſhare,*
And the ſum of all their ſhares $x + x + 72 + x + 172$.
or $3x + 244 = 1000$ *by the queſtion,*
that is $3x = 1000 - 244 = 756$,
or $x = \frac{756}{3} = 252$l. $=$ B's *ſhare,*
and $x + 72 = 252 + 72 = 324$l. $=$ A's *ſhare,*
and $x + 172 = 252 + 172 = 424$l. $=$ C's *ſhare.*

252l.
324l.
424l.

1000l. *the proof.*

4. A prize of 1000l. is to be divided between two perſons, whoſe ſhares therein are in the proportion of 7 to 9: required the ſhare of each.

Let $x =$ *firſt perſon's ſhare,*
then will $1000 - x =$ *ſecond perſon's ſhare,*
and $x : 1000 - x :: 7 : 9$, *by the queſtion,*

that is $9x=(1000-x)\times 7=7000-7x$,
or $9x+7x=16x=7000$,
whence $x=\frac{7000}{16}=437$l. 10s. = 1*ſt ſhare*,
and $1000-x=1000-437$l. 10s.$=562$l. 10s. *2d ſhare.*

5. The paving of a ſquare at 2s. a yard, coſts as much as the incloſing it at 5s. a yard: required the ſide of the ſquare.

Let $x=$ *ſide of the ſquare ſought,*
then $4x=$ *yards of incloſure,*
and $x^2=$ *yards of pavement;*
whence $4x\times 5=20x=$ *price of incloſing,*
and $x^2\times 2=2x^2=$ *price of paving.*
But $2x^2=20x$ *by the queſtion,*
therefore $x^2=10x$, *and* $x=10=$ *length of the ſide required.*

6. A labourer engaged to ſerve for 40 days upon theſe conditions, that for every day he worked he was to receive 20d. but for every day he played, or was abſent, he was to forfeit 8d. now at the end of the time he had to receive 1l. 11s. 8d. it is required to find how many days he worked, and how many he was idle.

Let x *be the number of days he worked,*
then will $40-x$ *be the number of days he was idle,*
alſo $x\times 20=20x=$ *ſum earned,*
and $(40-x)\times 8=320-8x=$ *ſum forfeited,*
whence $20x-(320-8x)=380d.$ $(=1l.\ 11s.\ 8d.)$ *by the queſtion, that is* $20x-320+8x=380$
or $28x=380+320=700$,
and $x=\frac{700}{28}=25=$ *number of days he worked,*
and $40-x=40-25=15=$ *number of days he was idle.*

7. Out of a caſk of wine, which had leaked away $\frac{1}{3}$, 21 gallons were drawn; and then, being gauged, it appeared to be half full: how much did it hold?

Let it be ſuppoſed to have held x gallons,

Then it would have leaked $\frac{x}{3}$ *gallons,*

and conſequently there had been taken away $21+\frac{x}{3}$ *gallons.*

But $21+\frac{x}{3}=\frac{x}{2}$ *by the queſtion,*

That is $63+x=\frac{3x}{2}$,

or $126+2x=3x$
hence $3x-2x=126$
or $x=126=$ *number of gallons required.*

8. What fraction is that to the numerator of which, if 1 be added, the value will be $\frac{1}{3}$; but, if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction be repreſented by $\frac{x}{y}$,

Then will $\frac{x+1}{y}=\frac{1}{3}$,

and $\frac{x}{y+1}=\frac{1}{4}$,

or $3x+3=y$,
and $4x=y+1$,
hence $4x-3x-3=y+1-y$
that is $x-3=1$
or $x=4$, *and* $y=3x+3$
$=12+3=15$

So that $\frac{4}{15}=$ *fraction required.*

9. A market-woman bought in a certain number of eggs at 2 a penny, and as many at 3 a penny, and ſold them all out again at the rate of 5 for twopence, and by ſo doing loſt 4d. what number of eggs had ſhe?

Let $x =$ *number of eggs of each ſort,*

Then will $\frac{x}{2} =$ *price of the firſt ſort,*

and $\frac{x}{3} =$ *price of the ſecond ſort.*

But $5 : 2 :: 2x$ (*the whole number of eggs*) $: \frac{4x}{5}$;

therefore $\frac{4x}{5}$ *price of both ſorts together, at five for 2d.*

And $\frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4$ *by the queſtion;*

that is $x + \frac{2x}{3} - \frac{8x}{5} = 8$;

or $3x + 2x - \frac{24x}{5} = 24$;

or $15x + 10x - 24x = 120$

whence $x = 120 =$ *number of eggs of each ſort required.*

10. If A can do a piece of work alone in ten days, and B in thirteen; ſet them both about it together, in what time will it be finiſhed?

Let the time ſought be denoted by x,

Then 10 *days* : 1 *work* :: x *days* : $\frac{x}{10}$,

and 13 *days* : 1 *work* :: x *days* : $\frac{x}{13}$,

hence $\frac{x}{10} =$ *part done by* A *in* x *days;*

and $\frac{x}{13} =$ *part done by* B *in* x *days.*

Conſequently $\frac{x}{10}+\frac{x}{13}=1$;

That is $\frac{13x}{10}+x=13$, *or* $13x+10x=130$;

And therefore $23x=130$, *or* $x=\frac{130}{23}=5\frac{15}{23}$ *days, the time required.*

11. If one agent A, alone, can produce an effect *e*, in the time *a*, and another agent B, alone, in the time *b*; in what time will they both together produce the ſame effect?

Let the time ſought be denoted by x,

Then $a:e::x:\frac{ex}{a}=$ *part of the effect produced by* A,

and $b:e::x:\frac{ex}{b}=$ *part of the effect produced by* B,

whence $\frac{ex}{a}+\frac{ex}{b}=e$ *by the queſtion;*

or $\frac{x}{a}+\frac{x}{b}=1$;

that is $x+\frac{ax}{b}=a$;

or $bx+ax=ba$;

and conſequently $x=\frac{ba}{b+a}=$ *time required.*

QUESTIONS FOR PRACTICE.

1. What two numbers are thoſe whoſe difference is 7, and ſum 33. *Anſ.* 13 *and* 20.

2. To divide the number 75 into two ſuch parts, that three times the greater may exceed ſeven times the leſs by 15. *Anſ.* 54 *and* 21.

3. In a mixture of wine and cyder, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons was cyder: how many gallons were there of each? *Anſ.* 85 *of wine, and* 35 *of cyder.*

4. A bill of 120l. was paid in guineas and moidores, and the number of pieces of both ſorts that were uſed was juſt 100; how many were there of each? *Anſ.* 50 *of each.*

5. Two travellers ſet out at the ſame time from London and York, whoſe diſtance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet? *Anſ. in* 10 *days.*

6. At a certain election 375 perſons voted, and the candidate choſen had a majority of 91; how many voted for each? *Anſ.* 233 *for one, and* 142 *for the other.*

7. What number is that from which, if 5 be ſubtracted, $\frac{2}{3}$ of the remainder will be 40? *Anſ.* 65.

8. A poſt is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length? *Anſ.* 24 *feet.*

9. There is a fiſh whoſe tail weighs 9℔. his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fiſh? *Anſ.* 72℔.

10. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purſe; what was in it at firſt? *Anſ.* 120 *guineas.*

11. A's age is double of B's, and B's is triple of C's, and the ſum of all their ages is 140; what is the age of each? *Anſ.* A's=84, B's=42, *and* C's=14.

12. Two perſons, A and B, lay out equal ſums of money in trade; A gains 126l. and B loſes 87l. and A's money is now double of B's; what did each lay out? *Anſ.* 300.

13. A perſon bought a chaiſe, horſe and harneſs, for 60l. the horſe came to twice the price of the harneſs, and the chaiſe to twice the price of the horſe and harneſs; what did he give for each?

Anſ. 13*l.* 6*s.* 8*d. for the horſe,* 6*l.* 13*s.* 4*d. for the harneſs, and* 40*l. for the chaiſe.*

14. Two perſons, A and B, have both the ſame income; A ſaves $\frac{1}{5}$ of his yearly, but B, by ſpending 50l. *per annum* more than A, at the end of 4 years finds himſelf 100l. in debt; what is their income?

Anſ. 125 *l.*

15. A gentleman has two horſes and a ſaddle, worth 50l. now if the ſaddle be put on the back of the firſt horſe, it will make his value double that of the ſecond; but if it be put on the back of the ſecond, it will make his value triple that of the firſt; what is the value of each horſe?

Anſ. One 30*l. and the other* 40*l.*

16. To divide the number 36 into three ſuch parts that $\frac{1}{2}$ of the firſt, $\frac{1}{3}$ of the ſecond, and $\frac{1}{4}$ of the third, may be all equal to each other?

Anſ. The parts are 8, 12, *and* 16.

17. A footman agreed to ſerve his maſter for 8l. a year and a livery, but was turned away at the end of 7 months, and received only 2l. 13s. 4d. and his livery; what was its value? *Anſ.* 4*l.* 16*s.*

18. A gentleman was deſirous of giving 3d. a-piece to ſome poor beggars, but found that he had not money enough in his pocket by 8d. he therefore gave them each 2d. and had then 3d. remaining; required the number of beggars? *Anſ.* 11.

19. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's three; but two of the greyhound's leaps are as much as three of the hare's; how many leaps muſt the greyhound take to catch the hare? *Anſ.* 300.

20. A perſon in play loſt $\frac{1}{4}$ of his money, and then won 3 ſhillings; after which he loſt $\frac{1}{3}$ of what he then had, and then won 2 ſhillings; laſtly he loſt $\frac{1}{7}$ of what he then had, and, this done, found he had but 12 s. remaining; what had he at firſt. *Anſ.* 20*s.*

21. To divide the number 90 into 4 ſuch parts, that if the firſt be increaſed by 2, the ſecond diminiſhed by 2, the third multiplied by 2, and the fourth divided by 2; the ſum, difference, product, and quotient ſhall be all equal to each other.

Anſ. The parts are 18, 22, 10, *and* 40, *reſpectively.*

22. The hour and minute hand of a clock are exactly together at 12 o'clock, when are they next together? *Anſ.* 1 *ho.* $5\frac{5}{11}$ *min.*

23. There is an iſland 73 miles in circumference, and three footmen all ſtart together to travel the ſame way about it: A goes 5 miles a day, B 8, and C 10; when will they all come together again? *Anſ.* 73 *days.*

24. How much foreign brandy at 8s. *per* gallon, and Britiſh ſpirits at 3s. *per* gallon, muſt be mixed together, ſo that in ſelling the compound at 9s. *per* gallon, the diſtiller may clear 30 *per cent.*?

Anſ. 51 *gallons of brandy, and* 14 *of ſpirits.*

25. A man and his wife uſually drank out a caſk of beer in 12 days; but when the man was from home it laſted the woman 30 days: how many days would the man alone be in drinking it? *Anſ.* 20 *days.*

26. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days: how many days will it take each perſon to perform the ſame work alone?

Anſ. A $14\frac{34}{49}$ *days,* B $17\frac{23}{41}$, *and* C $23\frac{7}{31}$.

27. If three agents, A, B, and C, can produce the effects *a*, *b*, *c*, in the times *e*, *f*, *g*, reſpectively; in what time would they jointly produce the effect *d*?

Anſ. $d \div \left(\frac{a}{e} + \frac{b}{f} + \frac{c}{g}\right)$ *time.*

QUADRATIC EQUATIONS.

A ſimple quadratic equation is that which involves the ſquare of the unknown quantity only.

An affected quadratic equation is that which involves the ſquare of the unknown quantity, together with the product that ariſes from multiplying it by ſome known quantity.

Thus $ax^2 = b$, *is a ſimple quadratic equation,*

And $ax^2 + bx = c$, *is an affected quadratic equation.*

The rule for a ſimple quadratic equation has been given already.

All affected quadratic equations fall under the three following forms:

1. $x^2 + ax = b$
2. $x^2 - ax = b$
3. $x^2 - ax = -b$.

The rule for finding the value of x, in each of theſe equations, is as follows:

RULE [*].

1. Tranſpoſe all the terms that involve the unknown quantity to one ſide of the equation, and the known terms to the other, and let them be ranged according to their dimenſions.

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[*] The ſquare root of any quantity may be either + or —, and therefore all quadratic equations admit of two ſolutions. Thus the ſquare root of $+n^2$ is $+n$ or $-n$; for $+n \times +n$ or $-n \times -n$ are each equal to $+n^2$.

2. When the ſquare of the unknown quantity has any co-efficient prefixed to it, let all the reſt of the terms be divided by that co-efficient.

3. Add the ſquare of half the co-efficient of the ſecond term to both ſides of the equation, and that ſide which involves the unknown quantity will then be a complete ſquare.

4. Extract the ſquare root from both ſides of the equation, and the value of the unknown quantity will be determined, as was required.

Note, 1. The ſquare root of one ſide of the equation is always equal to the unknown quantity, with half the co-efficient of the ſecond term ſubjoined to it.

2. All equations, in which there are two terms involving the unknown quantity, and which have the index of the one juſt double that of the other, are ſolved like quadratics, by completing the ſquare.

Thus, $x^4+ax^2=b$, *or* $x^n+ax^{\frac{n}{2}}=b$, *are the ſame as quadratics, and the value of the unknown quantity may be determined accordingly.*

So, in the firſt form, $(x^2+ax=b)$ where $x+\frac{1}{2}a$ is found $=\surd(b+\frac{1}{4}a^2)$ the root may be either $+\surd(b+\frac{1}{4}a^2)$ or $-\surd(b+\frac{1}{4}a^2)$ ſince either of them being multiplied by itſelf will produce Square $b+\frac{1}{4}a^2$. And this ambiguity is expreſſed by writing the uncertain ſign ± before $\surd(b+\frac{1}{4}a^2)$; thus $x=\pm\surd(b+\frac{1}{4}a^2)-\frac{1}{2}a$.

In this form, where $x=\pm\surd(b+\frac{1}{4}a^2)-\frac{1}{2}a$, the firſt value of x, viz. $x=+\surd(b+\frac{1}{4}a^2)-\frac{1}{2}a$ is always affirmative; for ſince $\frac{1}{4}a^2+b$ is greater than $\frac{1}{4}a^2$, the greateſt ſquare muſt neceſſarily have the greateſt ſquare root; therefore $\surd(b+\frac{1}{4}a^2)$ will always be greater than $\surd\frac{1}{4}a^2$, or its equal $\frac{1}{2}a$; and conſequently $+\surd(b+\frac{1}{4}a^2)-\frac{1}{2}a$ will always be affirmative.

EXAMPLES.

1. Given $x^2+4x=140$ to find x.

Firſt, $x^2+4x+4=140+4=144$, *by completing the ſquare;*
then $\surd(x^2+4x+4)=\surd 144$ *by extracting the root;*
or, which is the ſame thing, $x+2=12$,
and therefore $x=12-2=10$.

2. Given $x^2-6x+8=80$, to find x.

Firſt, $x^2-6x=80-8=72$ *by tranſpoſition;*
Then $x^2-6x+9=72+9=81$ *by completing the ſquare;*
And $x-3=\surd 81=9$ *by extracting the root;*
Therefore $x=9+3=12$.

The ſecond value, viz. $x=-\surd(b+\frac{1}{4}a^2)-\frac{1}{2}a$ will always be negative, becauſe it is compoſed of two negative terms. Therefore when $x^2+ax=b$, we ſhall have $x=+\surd(b+\frac{1}{4}a^2)-\frac{1}{2}a$ for the affirmative value of x, and $x=-\surd(b+\frac{1}{4}a^2)-\frac{1}{2}a$ for the negative value of x.

In the ſecond form, where $x=\pm\surd(b+\frac{1}{4}a^2)+\frac{1}{2}a$ the firſt value, viz. $x=+\surd(b+\frac{1}{4}a^2)+\frac{1}{2}a$ is always affirmative, ſince it is compoſed of two affirmative terms. The ſecond value, viz. $x=-\surd(b+\frac{1}{4}a^2)+\frac{1}{2}a$ will always be negative; for ſince $b+\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2$, the ſquare root of $b+\frac{1}{4}a^2$ $(\surd b+\frac{1}{4}a^2)$ or will be greater than $\surd\frac{1}{4}a^2$, or it's equal $\frac{1}{2}a$; and conſequently $-\surd(b+\frac{1}{4}a^2)+\frac{1}{2}a$ is always a negative quantity.

Therefore when $x^2-ax=b$, we ſhall have $x=+\surd(b+\frac{1}{4}a^2)+\frac{1}{2}a$ for the affirmative value of x, and $x=-\surd(b+\frac{1}{4}a^2)+\frac{1}{2}a$ for the negative value of x; ſo that in both the firſt and ſecond forms, the unknown quantity has always two values, one of which is poſitive, and the other negative.

3. Given $2x^2+8x-20=70$, to find x.

Firſt, $2x^2+8x=70+20=90$ *by tranſpoſition.*
Then $x^2+4x=45$ *by dividing by* 2,
And $x^2+4x+4=49$ *by completing the ſquare;*
Whence $x+2=\sqrt{49}=7$ *by extracting the root.*
And conſequently $x=7-2=5$.

4. Given $3x^2-3x+6=5\frac{1}{3}$, to find x.

Here, $x^2-x+2=1\frac{7}{9}$ *by dividing by* 3,
And $x^2-x=1\frac{7}{9}-2$ *by tranſpoſition;*
Alſo $x^2-x+\frac{1}{4}=1\frac{7}{9}-2+\frac{1}{4}=\frac{1}{36}$ *by completing the ſqu.*
And $x-\frac{1}{2}=\sqrt{\frac{1}{36}}=\frac{1}{6}$ *by evolution;*
Therefore $x=\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$ *the anſwer.*

5. Given $\dfrac{x^2}{2}-\dfrac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$, to find x.

Here $\dfrac{x^2}{2}-\dfrac{x}{3}=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$ *by tranſpoſition,*

And $x^2-\dfrac{2x}{3}=44\frac{1}{3}$ *by dividing by* $\frac{1}{2}$;

Whence $x^2-\dfrac{2x}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=44\frac{4}{9}$ *by completing the ſquare,*
And $x-\frac{1}{3}=\sqrt{44\frac{4}{9}}=6\frac{2}{3}$ *by evolution.*
Therefore $x=6\frac{2}{3}+\frac{1}{3}=7$ *the anſwer.*

In the third form, where $x=\pm\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$, both the values of x will be poſitive, ſuppoſing $\frac{1}{4}a^2$ is greater than b. For the firſt value, viz. $x=+\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ is evidently affirmative, being compoſed of two affirmative terms.

The 2d value, viz. $x=-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ is alſo affirmative; for ſince $\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2-b$, therefore $\sqrt{\frac{1}{4}a^2}$ or $\frac{1}{2}a$ is greater than $\sqrt{(\frac{1}{4}a^2-b)}$, and conſequently $-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ will always be an affirmative quantity. Therefore, when $x^2-ax=-b$, we ſhall have $x=+\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ and alſo $x=-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ for the affirmative values of x.

6. Given $ax^2+bx=c$, to find x.

Firſt, $x^2+\frac{b}{a}x=\frac{c}{a}$ *by diviſion;*

Then $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}$ *by completing the ſquare;*

And $x+\frac{b}{2a}=\surd(\frac{c}{a}+\frac{b^2}{4a^2})=\surd(\frac{4ac+b^2}{4a^2})$ *by evolution;*

Therefore $x=\pm\surd(\frac{4ac+b^2}{4a^2})-\frac{b}{2a}$.

7. Given $ax^2-bx+c=d$, to find x.

Here, $ax^2-bx=d-c$ *by tranſpoſition.*

And $x^2-\frac{b}{a}x=\frac{d-c}{a}$ *by diviſion.*

Alſo $x^2-\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{d-c}{a}+\frac{b^2}{4a^2}$ *by completing the ſquare;*

And $x-\frac{b}{2a}=\pm\surd(\frac{d-c}{a}+\frac{b^2}{4a^2})$ *by evolution;*

Therefore $x=\frac{b}{2a}\pm\surd(\frac{d-c}{a}+\frac{b^2}{4a^2})$.

8. Given $x^4+2ax^2=b$, to find x.

Here, $x^4+2ax^2+a^2=b+a^2$ *by completing the ſquare,*
And $x^2+a=\surd(b+a^2)$ *by evolution;*
Whence $x^2=\surd(b+a^2)-a$.
And conſequently $x=\surd(\surd(b+a^2)-a)$.

But in this third form, if b be greater than $\frac{1}{4}a^2$ the ſolution of the propoſed queſtion will be impoſſible. For ſince the ſquare of any quantity (whether that quantity be affirmative or negative) is always affirmative, the ſquare root of a negative quantity is impoſſible, and cannot be aſſigned. But if b be greater than $\frac{1}{4}a^2$, then $\frac{1}{4}a^2-b$ is a negative quantity; and therefore $\surd(\frac{1}{4}a^2-b)$ is impoſſible, or only imaginary; conſequently in that caſe $x=\frac{1}{2}a\pm\surd(\frac{1}{4}a^2-b)$ is alſo impoſſible or imaginary.

9. Given $ax^n - bx^{\frac{n}{2}} - c = -d$, to find x.

Firſt, $ax^n - bx^{\frac{n}{2}} = c - d$ *by tranſpoſition,*

And $x^n - \frac{b}{a}x^{\frac{n}{2}} = \frac{c-d}{a}$ *by diviſion,*

Alſo $x^n - \frac{b}{a}x^{\frac{n}{2}} + \frac{b^2}{4a^2} = \frac{c-d}{a} + \frac{b^2}{4a^2}$ *by completing the ſqu.*

And $x^{\frac{n}{2}} - \frac{b}{2a} = \sqrt{(\frac{c-d}{a} + \frac{b^2}{2a^2})}$ *by evolution;*

Therefore $x^{\frac{n}{2}} = \frac{b}{2a} \pm \sqrt{(\frac{c-d}{a} + \frac{b^2}{4a^2})}$,

And conſequently $x = (\frac{b}{2a} \pm \sqrt{\frac{4ac - 4ad + b^2}{4a^2}})^{\frac{2}{n}}$.

EXAMPLES FOR PRACTICE.

1. Given $x^2 - 8x + 10 = 19$ to find x. *Anſ.* $x = 9$.

2. Given $x^2 - x - 40 = 170$ to find x. *Anſ.* $x = 15$.

3. Given $3x^2 + 2x - 9 = 76$ to find x. *Anſ.* $x = 5$.

4. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{1}{5} = 20$ to find x.
Anſ. $x = 5.4039$.

5. Given $x^2 + x = a$ to find x. *Anſ.* $x = \sqrt{(a + \frac{1}{4})} - \frac{1}{2}$.

6. Given $x^2 + ax - b = c$ to find x.
Anſ. $x = \sqrt{(c + b + \frac{1}{4}a^2)} - \frac{1}{2}a$.

7. Given $x^2 - ax = -b$ to find x.
Anſ. $x = \pm\sqrt{(\frac{1}{4}a^2 - b)} + \frac{1}{2}a$,

8. Given $\frac{2}{3}x^2 + \frac{3}{4}x = \frac{4}{5}$ to find x. *Anſ.* 1.7939.

9. Given $2x^4 - x^2 + 104 = 600$ to find x. *Anſ.* $x = 4$.

10. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$ to find the value of x.
Anſ. $x = 49$.

11. Given $3x^n - 2x^{\frac{n}{2}} - 5 = 20$ to find x.
Anſ. $x = (\frac{1}{3}\sqrt{76} + \frac{1}{3})^{\frac{2}{n}}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whoſe difference is 8, and product 240.

Let $x =$ *to the leaſt number,*
then will $x + 8 =$ *to the greater,*
and $x \times (x+8) = x^2 + 8x = 240$ *by the queſtion;*
Whence $x^2 + 8x + 16 = 240 + 16 = 256$ *by completing the ſquare,*
alſo $x + 4 = \sqrt{256} = 16$ *by evolution;*
And therefore $x = 16 - 4 = 12 =$ *leſſer number, and* $12 + 8 = 20 =$ *greater.*

2. To divide the number 60 into two ſuch parts that their product may be 864.

Let $x =$ *greater part,*
then will $60 - x =$ *leſſer,*
and $x \times (60 - x) = 60x - x^2 = 864$ *by the queſtion,*
that is $x^2 - 60x = -864$;
Whence $x^2 - 60x + 900 = -864 + 900 = 36$ *by completing the ſquare;*
Alſo $x - 30 = \sqrt{36} = 6$ *by extracting the root;*
and therefore $x = 6 + 30 = 36 =$ *greater part,*
and $60 - x = 60 - 36 = 24$ *leſſer.*

3. Given the ſum of two numbers $= 10$ (a), and the ſum of their ſquares $= 58$ (b); to find thoſe numbers.

Let $x =$ *greater of the two numbers,*
then will $a - x =$ *leſſer;*
And $x^2 + (a-x)^2 = 2x^2 + a^2 - 2ax = b$ *by the queſtion,*
or $x^2 + \frac{a^2}{2} - ax = \frac{b}{2}$ *by diviſion,*

or $x^2-ax=\frac{b}{2}-\frac{a^2}{2}=\frac{b-a^2}{2}$ *by tranſpoſition.*

Whence $x^2-ax+\frac{a^2}{4}=\frac{b-a^2}{2}+\frac{a^2}{4}=\frac{2b-a^2}{4}$ *by completing the ſquare.*

Alſo $x-\frac{a}{2}=\sqrt{\frac{2b-a^2}{4}}$ *by extracting the root ;*

And therefore $x=\pm\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ *greater number,*

And $a-(\frac{a}{2}\pm\sqrt{\frac{2b-a^2}{4}})=\mp\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ *leſſer.*

Hence theſe two theorems, being put into numbers, give 7 *and* 3, *for the numbers required.*

4. Sold a piece of cloth for 24l. and gained as much *per cent.* as the cloth coſt me; what was the price of the cloth?

Let $x=$ *pounds the cloth coſt,*
Then $24-x=$ *whole gain,*
But $100 : x :: x : 24-x$ *by the queſtion,*
Or $x^2=100\times(24-x)=2400-100x$,
That is $x^2+100x=2400$,
Whence $x^2+100x+2500=2400+2500=4900$ *by completing the ſquare,*
And $x+50=\sqrt{4900}=70$ *by extraction of roots,*
Conſequently $x=70-50=20=$ *price of the cloth.*

5. A perſon bought a number of oxen for 80l. and if he had bought four more for the ſame money, he would have paid 1l. leſs for each; how many did he buy?

Let the number of oxen be repreſented by x,
Then will $\frac{80}{x}$ *be the price of each,*

And $\frac{80}{x+4}=$ *price of each if* $x+4$ *had coſt* 80*l.*

But $\frac{80}{x}=\frac{80}{x+4}+1$ *by the queſtion,*

Or $80=\frac{80x}{x+4}+x$,

Or $80x+320=80x+x^2+4x$,

That is $x^2+4x=320$,

Whence $x^2+4x+4=320+4=324$ *by completing the ſquare,*

And $x+2=\sqrt{324}=18$ *by evolution,*

Conſequently $x=18-2=16=$ *number of oxen required.*

6. What two numbers are thoſe whoſe ſum, product, and difference of their ſquares, are all equal to each other?

Let $x=$ *greater number,*

And $y=$ *leſſer,*

Then $\left.\begin{cases} x+y=xy \\ x+y=x^2-y^2 \end{cases}\right\}$ *by the queſtion,*

And $1=\frac{x^2-y^2}{x+y}=x-y$, *or* $x=y+1$ *from the 2d equation.*

Alſo $(y+1)+y=(y+1)\times y$ *from the firſt equation,*

Or $2y+1=y^2+y$,

That is $y^2-y=1$,

Whence $y^2-y+\frac{1}{4}=1\frac{1}{4}$ *by completing the ſquare,*

Alſo $y-\frac{1}{2}=\sqrt{1\frac{1}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$ *by evolution,*

Conſequently $y=\frac{\sqrt{5}}{2}+\frac{1}{2}=\frac{\sqrt{5}+1}{2}$, *and* $x=y+1=\frac{\sqrt{5}+3}{2}$. *And if theſe expreſſions be turned into numbers,*

we ſhall have $x=2.6180+$

and $y=1.6180+$

7. There are four numbers in arithmetical progreſſion, of which the product of the two extremes is 45, and that of the means 77; what are the numbers?

Let $x =$ *leſſer extreme,*
and $y =$ *common difference,*
Then x, $x+y$, $x+2y$, $x+3y$, *will be the four numbers,*

$$\text{and} \left\{\begin{array}{l} x \times (x+3y) = x^2+3xy = 45 \\ (x+y) \times (x+2y) = x^2+3xy+2y^2 = 77 \end{array}\right\} \text{by the queſtion,}$$

Whence $2y^2 = 77 - 45 = 32$, *and* $y^2 = \frac{32}{2} = 16$ *by ſubtraction and diviſion,*
Or $y = \sqrt{16} = 4$ *by evolution,*
Therefore $x^2+3xy = x^2+12x = 45$ *by the 1ſt equation,*
Alſo $x^2+12x+36 = 45+36 = 81$ *by completing the ſquare,*
And $x+6 = \sqrt{81} = 9$ *by the extraction of roots,*
Conſequently $x = 9-6 = 3$, *and the numbers are* 3, 7, 11, *and* 15.

8. To find three numbers in geometrical progreſſion, whoſe ſum ſhall be 14, and the ſum of their ſquares 84.

Let x, y, *and* z, *be the numbers ſought,*
Then $xz = y^2$ *by the nature of proportion,*

$$\text{and} \left\{\begin{array}{l} x+y+z = 14 \\ x^2+y^2+z^2 = 84 \end{array}\right\} \text{by the queſtion,}$$

But $x+z = 14-y$ *by the 2d equation,*
and $x^2+2xz+z^2 = 196-28y+y^2$ *by ſquaring both ſides,*
Or $x^2+z^2+2y^2 = 196-28y+y^2$ *by putting* $2y^2$ *for its equal* $2xz$,
That is $x^2+z^2+y^2 = 196-28y$ *by ſubtraction,*
or $196-28y = 84$ *by equality,*
Hence $y = \frac{196-84}{28} = 4$ *by tranſpoſition and diviſion.*

Again, $xz = y^2 = 16$, *or* $x = \frac{16}{z}$ *by the 1ſt equation,*

And $x+y+z = \frac{16}{z}+4+z = 14$ *by the 2d equation,*

Or $16+4z+z^2 = 14z$, *or* $z^2-10z = -16$,

Whence $z^2-10z+25 = 25-16 = 9$ *by completing the ſquare,*

And $z-5 = \sqrt{9} = 3$, *or* $z = 3+5 = 8$,

Conſequently $x = 14-y-z = 14-4-8 = 2$, *and the numbers are* 2, 4, 8.

9. The ſum (s) and the product (p) of any two numbers being given; to find the ſum of the ſquares, cubes, biquadrates, &c. of thoſe numbers.

Let the two numbers be denoted by x *and* y.

Then will $\left\{\begin{matrix} x+y=s \\ xy=p \end{matrix}\right\}$ *by the queſtion,*

But $(x+y)^2 = x^2+2xy+y^2 = s^2$ *by involution,*

and $x^2+2xy+y^2-2xy = s^2-2p$ *by ſubtraction.*

that is $x^2+y^2 = s^2-2p =$ *ſum of the ſquares.*

Again, $(x^2+y^2)\times(x+y) = (s^2-2p)\times s$ *by multiplication,*

or $x^3+xy\times(x+y)+y^3 = s^3-2sp$,

or $x^3+sp+y^3 = s^3-2sp$ *by ſubſtituting* sp *for its equal* $xy\times(x+y)$;

and therefore $x^3+y^3 = s^3-3sp =$ *ſum of the cubes.*

In like manner, $(x^3+y^3)\times(x+y) = (s^3-3sp)\times s$ *by multiplication,*

or $x^4+xy\times(x^2+y^2)+y^4 = s^4-3s^2p$,

or $x^4+p\times(s^2-2p)+y^4 = s^4-3s^2p$ *by ſubſtituting* $p\times(s^2-2p)$ *for its equal* $xy\times(x^2+y^2)$;

And conſequently, $x^4+y^4 = s^4-3s^2p-p\times(s^2-2p) = s^4-4s^2p+2p^2 =$ *ſum of the biquadrates, or fourth powers;*

And the ſum of the nth. powers is $s^n - ns^{n-2}p + n.\frac{n-3}{2}.s^{n-4}p^2 - n.\frac{n-4}{2}.\frac{n-5}{3}.s^{n-6}p^3 + n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}.s^{n-8}p^4$, &c.

10. The ſum (a) and the ſum of the ſquares (b) of four numbers, in geometrical progreſſion, being given; to find thoſe numbers.

Let x *and* y *denote the two means,*

then will $\frac{x^2}{y}$ *and* $\frac{y^2}{x}$ *be the two extremes by the nature of proportion.*

Alſo, let the ſum of the two means $=s$, *and their product* $=p$.

Then will the ſum of the two extremes $=a-s$ *by the queſtion, and their product* $=p$, *by the nature of proportion.*

$$\textit{Hence} \left\{\begin{array}{l} x^2+y^2=s^2-2p \\ \frac{x^4}{y^4}+\frac{y^4}{x^2}=(a-s)^2-2p \end{array}\right\} \textit{by the laſt problem,}$$

And $x^2+y^2+\frac{x^4}{y^2}+\frac{y^4}{x^2}=s^2+(a-s)^2-4p=b$ *by the queſtion.*

Again, $\frac{x^2}{y}+\frac{y^2}{x}=a-s$ *by the queſtion,*

or $x^3+y^3=xy\times(a-s)=p\times(a-s)$.

But $x^3+y^3=s^3-3sp$ *by the laſt problem;*

and therefore $p\times(a-s)=s^3-3sp$ *by equality,*

or $pa-ps+3ps=pa+2ps=s^3$,

or $p=\frac{s^3}{a+2s}$ *by diviſion;*

Whence $s^2+(a-s)^2-4p=s^2+(a-s)^2-\frac{4s^3}{a+2s}=b$, *by ſubſtitution,*

or $s^2+\frac{b}{a}s=\frac{a^2-b}{2}$ *by reduction.*

And $s=\sqrt{\left(\frac{a^2-b}{2}+\frac{b^2}{4a^2}\right)}-\frac{b}{2a}$ *by comp. the ſquare, and extracting the root.*

And from this value of s, *all the reſt of the quantities* p, x, *and* y, *may be readily determined.*

QUESTIONS FOR PRACTICE.

1. What two numbers are thoſe whoſe ſum is 20, and their product 36? *Anſ.* 2 *and* 18.

2. To divide the number 60 into two ſuch parts, that their product may be to the ſum of their ſquares in the ratio of 2 to 5. *Anſ.* 20 *and* 40.

3. The difference of two numbers is 3, and the difference of their cubes is 117: what are thoſe numbers? *Anſ.* 2 *and* 5.

4. A company at a tavern had 8l. 15s to pay for their reckoning; but, before the bill was ſettled, two of them ſneaked off, and then thoſe who remained had 10s. a-piece more to pay than before: how many were there in company? *Anſ.* 7.

6. A grazier bought as many ſheep as coſt him 60l. and, after reſerving 15 out of the number, he ſold the remainder for 54l. and gained 2s. a-head by them; how many ſheep did he buy? *Anſ.* 75.

7. There are two numbers whoſe difference is 15, and half their product is equal to the cube of the leſſer number; what are thoſe numbers? *Anſ.* 3 *and* 18.

8. A perſon bought cloth for 33l. 15s. which he ſold again at 2l. 8s. *per* piece, and gained by the bargain as much as one piece coſt him; required the number of pieces? *Anſ.* 15.

9. What number is that, which, when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? *Anſ.* 24.

10. What two numbers are thoſe whoſe ſum multiplied by the greater is equal to 77; and whoſe difference multiplied by the leſſer is equal to 12? *Anſ.* 4 *and* 7.

11. To find a number ſuch that if you ſubtract it from 10, and multiply the remainder by the number itſelf, the product ſhall be 21. *Anſ. 7 or 3.*

12. When will the hour, minute, and ſecond-hands of a clock be all together next after 12 o'clock?

Anſ. Only at 12 o'clock.

13. The ſum of two numbers is 8, and the ſum of their cubes is 152; what are the numbers?

Anſ. 3 *and* 5.

14. The ſum of two numbers is 7, and the ſum of their 4th powers is 641; what are the numbers?

Anſ. 2 *and* 5.

15. The ſum of two numbers is 6, and the ſum of their 5th powers is 1056; what are the numbers?

Anſ. 2 *and* 4.

16. The ſum of four numbers in arithmetical progreſſion is 56, and the ſum of their ſquares is 864; what are the numbers? *Anſ.* 8, 12, 16, *and* 20.

17. To find four numbers in geometrical progreſſion whoſe ſum is 15, and the ſum of their ſquares 85?

Anſ. 1, 2, 4, *and* 8.

18. There are two numbers whoſe difference is 15, and half their product is equal to the cube of the leſſer number; what are thoſe numbers? *Anſ.* 3 *and* 18.

19. It is required to divide the number 24 into two ſuch parts, that their product may be equal to 35 times their difference. *Anſ.* 10 *and* 14.

20. It is required to find four numbers in arithmetical progreſſion, ſuch that their common difference may be 4, and their continued product 176985.

Anſ. 15, 19, 23 *and* 27.

21. Two partners, A and B, gained 140l. by trade; A's money was three months in trade, and his gain was 60l. leſs than his ſtock; and B's money, which was 50l. more than A's, was in trade five months: what was A's ſtock? *Anſ.* 100*l.*

OF THE NATURE AND FORMATION OF

EQUATIONS IN GENERAL.

All equations of ſuperior orders are generated by the multiplication of thoſe of inferior orders, involving the ſame unknown quantity.

Thus, *a quadratic equation* is formed by the multiplication of two ſimple equations.

A cubic equation is produced by the continued multiplication of three ſimple equations; or from one quadratic and one ſimple equation.

A biquadratic equation is generated by the continued multiplication of four ſimple equations; or of two quadratic equations; or of one cubic and one ſimple equation, &c.

For, ſuppoſe the unknown quantity to be x, and its values in any ſimple equation to be a, b, c, d, &c.

Then thoſe ſimple equations, by bringing all the terms to one ſide, will become $x-a=o$, $x-b=o$, $x-c=o$, $x-d=o$, &c.

And the product of any two of theſe, as $(x-a)\times(x-b)=o$, will form a *quadratic equation*, or one of two dimenſions.

The product of any three of them, as $(x-a)\times(x-b)\times(x-c)=o$, will form a *cubic equation*, or one of three dimenſions

The product of any four of them, as $(x-a)\times(x-b)\times(x-c)\times(x-d)=o$, will form a biquadratic equation, or one of four dimenſions, &c.

From hence it appears that every equation has as many roots as it has ſimple equations which produce it, or as there are units in the higheſt dimenſion of the unknown quantity.

For if any of the values of x (a, b, c, *or* d,) be ſubſtituted in the place of x in this biquadratic equation $(x-a)\times(x-b)\times(x-c)\times(x-d)$, all the terms of that equation will vaniſh, and the whole will be equal to nothing.

But as there are no other quantities, beſides theſe four, which, ſubſtituted in the place of x, will make the product vaniſh, it is plain that the equation cannot poſſibly have more than four roots, or admit of more than four ſolutions.

And, after the ſame manner, it may be ſhewn that no equation whatever can have more roots than it contains dimenſions of the unknown quantity.

To make this ſtill plainer by an example in numbers; ſuppoſe the equation to be reſolved be $x^4-10x^3+35x^2-50x+24=0$, and that you diſcover this equation to be the ſame with the product of $(x-1)\times(x-2)\times(x-3)\times(x-4.)$

Then you will certainly infer that the four values of x are 1, 2, 3, and 4; for any of theſe numbers being put for x will make that product, and conſequently $x^4-10x^3+35x^2-50x+24$, equal to nothing, according to the propoſed equation.

And it is certain that there can be no other values of x beſides theſe four; ſince, if any other number be ſubſtituted for x in thoſe factors, none of them will vaniſh, and therefore their product cannot be equal to nothing according to the equation.

The roots of equations are, alſo, either *poſitive* or *negative*, according as the roots of the ſimple equations from whence they are produced are poſitive or negative.

Thus, if you ſuppoſe $x=-a$, $x=b$, $x=-c$, and $x=d$, then will $x+a=0$, $x-b=0$, $x+c=0$, and $x-d$

$=0$, and the equation $(x+a)\times(x-b)\times(x+c)\times(x-d)=0$ will have its roots $-a$, $+b$, $-c$, $+d$.

The *ſigns* and *co-efficients* of equations will be beſt underſtood by conſidering the following table; where the ſimple equations $x-a$, $x-b$, &c. being multiplied continually together, produce, ſucceſſively, the higher equations.

$$\begin{array}{l}
x-a=0\\
x-b=0\\
\hline
\left.\begin{array}{l} x^2-ax\\ \quad -bx+ba\end{array}\right\}=0,\ \textit{a quadratic},\\
\hline
\times x-c=0\\
\hline
\left.\begin{array}{r} x^3-a\\ -b\\ -c\end{array}\right\} x^2 \left.\begin{array}{l} +ab\\ +ac\\ +bc\end{array}\right\} x-abc=0,\ \textit{a cubic},\\
\hline
\times x-d=0\\
\hline
\left.\begin{array}{r} x^4-a\\ -b\\ -c\\ -d\end{array}\right\} x^3 \left.\begin{array}{l} +ab\\ +ac\\ +ad\\ +bc\\ +bd\\ +cd\end{array}\right\} x^2 \left.\begin{array}{l} -abc\\ -abd\\ -acd\\ -bcd\end{array}\right\} x+abcd=0,\ \textit{a biquadratic}.\\
\hline
\end{array}$$

&c.

From the inſpection of theſe equations it is plain that the co-efficient of the firſt term is *unity*.

The co-efficient of the ſecond term is *the ſum of all the roots* $(a, b, c, d,)$ *with contrary ſigns*.

The co-efficient of the third term is *equal to the ſum of the rectangles of thoſe roots, or of all the products that can poſſibly ariſe by combining them two and two*.

The co-efficient of the fourth term is *equal to the ſum of all the products that can poſſibly ariſe by combining them, three and three*; and ſo on for any other co-efficient whatever.

The laſt term *is always equal to the product of all the roots with contrary ſigns*; and this reaſoning will
+ hold, whether the roots be poſitive or negative.

It likewiſe appears, from inſpection, that the ſigns of all the terms of any equation in the table are alternately + and —.

Thus the firſt term is always ſome pure power of x, and is poſitive.

The ſecond term is ſome power of x, multiplied by the quantities $-a$, $-b$, $-c$, &c. and, ſince theſe quantities are all negative, it follows that the term itſelf muſt be negative alſo.

The third term has the product of any two of theſe quantities ($-a$, $-b$, $-c$,) for its co-efficient, and is therefore poſitive; ſince $- \times -$, as well as $+ \times +$, gives, +, or an affirmative quantity.

For the ſame reaſon the next co-efficient, which is formed of the products of any three of theſe negative quantities, muſt be negative; and the next following, being made up of the products of any four of the ſaid negative quantities, muſt be poſitive; and ſo on.

And, from this reaſoning, it plainly appears that *when all the roots are poſitive, the ſigns are plus and minus alternately*.

But if the roots be all negative, as $x=-a$, $x=-b$, $x=-c$, $x=-d$, then $(x+a)\times(x+b)\times(x+c)\times(x+d)=0$, will expreſs the equation to be produced; and all the terms will plainly be poſitive.

So that, *when all the roots of an equation are negative, it is plain that there can be no changes in the ſigns of the terms of that equation.*

And, in general, there will be as many poſitive roots in any equation, as there are changes in the

+ If ye root is with the sign — this observatn will hold good, otherwise not.

ſigns of the terms of that equation, from + to −, or from − to +; and all the reſt of the roots will be negative.

From this rule it follows that, in quadratic equations, the two roots may be either both poſitive or both negative, or one negative and one affirmative.

Thus, in the equation, $x^2\begin{array}{l}-ax\\-bx\end{array}+ab=o$, or $(x-a)\times(x-b)$ there are two changes of the ſigns, and therefore the roots are both poſitive.

In the equation $x^2\begin{array}{l}+ax\\+bx\end{array}+ab,=o$, or $(x+a)\times(x+b)$ there is no change of the ſigns, and conſequently they are both negative.

And, in the equation $x^2\begin{array}{l}-ax\\+bx\end{array}-ab=o$, or $(x-a)\times(x+b)$ one of the roots will be affirmative and one negative; for, ſince the firſt term is poſitive and the laſt negative, there can be but one change in the ſigns, whether the ſecond term be + or −.

In cubic equations, the roots may be all poſitive, or all negative; or two of them may be negative and one poſitive, or one negative and two poſitive.

Thus, in the equation $(x-a)\times(x-b)\times(x-c)=o$, the ſigns will be alternately + and −; and, as the number of changes is three, the roots muſt be all poſitive.

In the equation $(x+a)\times(x+b)\times(x+c)=o$, where there are no changes of the ſigns, the roots muſt be all negative.

In the equation $x^3-(a+b-c)x^2+(ab-ac-bc)x+abc=o$, or $(x-a)\times(x-b)\times(x+c)$ the number of changes will be two, and conſequently two of the roots will be affirmative and one negative.

For if $a+b$ be greater than c, the ſecond term muſt be negative, its co-efficient being $-a$, $-b$, $+c$;

If $(a+b)$ is greater than c, $(-(a+b-c)$ is negative
& $ab-ac-bc$, may be either positive or negative
& in either of ye cases yre are 2 changes in ye equatn.
But if $a+b$ is less than c, $-(a+b-c)$ is positive; yn
$ab-ac-bc$, is necessarily neg. bec. $ab=a\times b$, &
$ac-bc=-c\times(a+b)$, but either of these 2 factors is greatr yn
& b; yn ye quanty $ac-bc$, wch is neg. is greatr than ab.

and if $a+b$ be leſs than c, the third term muſt be negative, its co-efficient $+ab-ac-bc$ being in that caſe negative.

In the equation $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc=0$, there can be only one change of the ſigns, and therefore one of the roots is poſitive, and the other two negative.

For if $a+b$ be leſs than c, then the ſecond term is negative, and the third muſt be negative alſo; and if $a+b$ be greater than c the ſecond term will be poſitive, and there can be but one change in the other two terms, whatever may be their ſigns.

And, in the ſame manner, this reaſoning may be extended to equations of higher dimenſions, and therefore the rule will be found to be true in all kinds of equations whatever.

PROBLEM I.

To increaſe or diminiſh the roots of an equation by any given quantity[*].

RULE.

1. Take ſome new letter, and connect it with the given quantity by the ſigns — or +, according as it is required to be increaſed or diminiſhed.

2. Subſtitute the powers of this quantity in the equation, inſtead of the powers of the unknown letter, and there will ariſe a new equation, whoſe roots will be augmented or diminiſhed as required.

---

[*] When a cubic equation has two equal roots, it may always be reduced to a lower dimenſion, and the ſolution, by that means, made more eaſy.

EXAMPLES.

1. Let the quadratic equation $x^2+8x+15=0$, be given; it is required to increaſe its roots by 7.

$$\begin{array}{rrrr} \textit{Suppoſe } x= & y-7, + & & \\ \textit{Then } x^2= & y^2 & -14y & +49 \\ 8x= & & +\ 8y & -56 \\ 15= & & & +15 \\ \hline & y^2 & -\ 6y & +\ 8=0= \textit{ to the equation required *.} \end{array}$$

2. Let $x^3-px^2+qx-r=0$, be the equation given; it is required to diminiſh the roots by the quantity e.

$$\left.\begin{array}{rrrrr} \textit{Suppoſe } x= & y+e; & & & \\ \textit{Then } x^3= & y^3 & +3ey^2 & +3e^2y & +e^3 \\ -px^2= & & -\ py^2 & -2pey & -pe^2 \\ +qx= & & & +\ qy & +qe \\ -\ r= & & & & -r \end{array}\right\} =0, \textit{ the new equation required †.}$$

3. Let $x^3+x^2-10x+8=0$ be given, and let its roots be increaſed by 4.

---

* For in the former equation $x^2+8x+15=0$, the roots are -3 and -5, and in the equation $y^2-6y+8=0$, the roots are 2 and 4; therefore the difference is 7, as was required.

† The laſt term of this transformed equation is the ſame as the equation given, having e in the place of y.

And from this it appears that, if the laſt term of any equation is to be deſtroyed, the difficulty will be no leſs than that of ſolving the original equation itſelf.

+ Let one of y^e roots of y^e proposed equat^n be supposed = to a, & b, y^e quant^y you desire to encrease by, then x must be =(a+b) that is x−b= a, but if to diminish by y^e same quant^y x, must be =(a−b) that is x+b=a &c.

Suppoſe $x=y-4$;

$$\begin{array}{rcrrrr}
\textit{Then } x^3 & = & y^3 & -12y^2 & +48y & -64 \\
+\ x^2 & = & + & y^2 & -\ 8y & +16 \\
-10x & = & & & -10y & +40 \\
+\ 8 & = & & & & +\ 8 \\
\hline
\end{array}$$

$$Sum = y^3 - 11y^2 + 30y = 0,$$

or $y^2 - 11y + 30 = 0$, *the equation required* *.

In which equation y *is found* $=6$; *and conſequently* $x=2$.

PROBLEM II.

To take away the ſecond term from any equation.

RULE. *

1. Divide the co-efficient of the ſecond term by the index of the higheſt power of the unknown quantity.

2 Annex the quotient, with its ſign changed, to ſome new letter, and this, being ſubſtituted for its equal in the given equation, will deſtroy the ſecond term, as required.

---

* In this example the given equation is reduced to a quadratic; and in the preſent caſe, as well as in all others, where the laſt term vaniſhes, the number aſſumed (-4) is one of the roots of the propoſed equation.

The affirmative roots of an equation are changed into negative ones of the ſame value, and the negative roots into affirmative ones, by only changing the ſigns of the terms alternately, beginning with the ſecond.

*

The reason of the above rule is as follows.

Let $x^2 \pm px \pm q = 0$; let $x = y + m$, Then by substit

$$\left.\begin{array}{rr} y^2 + 2my & + m^2 \\ \pm py & \pm pm \\ & \pm q \end{array}\right\} = 0;$$ Now in order to cancell y^e sec^d. Term, $2m \pm p = 0$

$2m = \mp p$; $m = \mp \frac{p}{2}$;

an Equat^n of y^e 3^d. degree is reducible to a 2^d. when

EXAMPLES.

1. Let the quadratic equation $x^2-8x+15=0$ be given; it is required to take away its ſecond term.

$$\begin{array}{rcl} \textit{Suppoſe}\ x & = & y+4\ (y+\tfrac{8}{2}); \\ \textit{Then}\ x^2 & = & y^2+8y+16 \\ -8x & = & -8y-32 \\ +15 & = & +15 \\ \hline & & y^2-1=0= \textit{equation required}^{*}. \end{array}$$

2. Let the equation $x^3-9x^2+26x-34=0$ be given; it is required to exterminate its ſecond term.

$$\begin{array}{rcl} \textit{Suppoſe}\ x & = & y+3\ (y+\tfrac{9}{3}); \\ \textit{Then}\ x^3 & = & y^3+9y^2+27y+27 \\ -9x^2 & = & -9y^2-54y-81 \\ +26x & = & +26y+78 \\ -34 & = & -34 \\ \hline & & y^3-y-10=0= \textit{equation required}. \end{array}$$

Thus the roots of the equation $x^4-x^3-19x^2+49x-30=0$, are $+1$, $+2$, $+3$, -5; but, by changing only the ſecond and fourth terms, the equation becomes $x^4+x^3-19x^2-49x-30=0$, and the roots are -1, -2, -3, $+4$.

All the roots of an equation may alſo be made affirmative or negative, by increaſing or diminiſhing each of them by ſome known quantity.

* From this example it appears that any quadratic equation may be reſolved without completing the ſquare, by only taking away the ſecond term; for ſince $y^2=1$, or $y=\sqrt{1}=1$, we ſhall have $x=y+4=1+4=5$, the root required. And the ſame may be ſhewn of any other adfected quadratic equation whatever.

ye last term is = 0, but ye last term is always of ye same form as ye propos'd equatn (v.107) therefore if it be = a, it is a proof yt what was added or subtracted from ye unknown quanty is one of ye roots of ye propd. equatn

3. Let $x^4+8x^3-5x^2+10x-4=0$ be given, to exterminate the ſecond term *.

Suppoſe $x=y-2\left(y-\frac{8}{4}\right)$;

$$\begin{array}{rrrrrr}
\textit{Then } x^4 = & y^4 & -8y^3 & +24y^2 & -32y & +16 \\
+8x^3 = & & +8y^3 & -48y^2 & +96y & -64 \\
-5x^2 = & & & -5y^2 & +20y & -20 \\
+10x = & & & & +10y & -20 \\
-4 = & & & & & -4 \\
\hline
 & y^4 & & -29y^2 & +94y & -92=0=\textit{equation required.}
\end{array}$$

4. Let $x^4-px^3+qx^2-rx+s=0$ be given, to exterminate the ſecond term.

Suppoſe $x=y+\frac{p}{4}$;

$$\begin{array}{rrrrrr}
\textit{Then } x^4 = & y^4 & +py^3 & +\frac{3p^2y^2}{8} & +\frac{p^3y}{16} & +\frac{p^4}{256} \\
-px^3 = & & -py^3 & -\frac{3p^2y^2}{4} & -\frac{3p^3y}{16} & -\frac{p^4}{64} \\
+qx^2 = & & & +qy^2 & +\frac{pqy}{2} & +\frac{qp^2}{16} \\
-rs = & & & & -ry & -\frac{rp}{4} \\
+s = & & & & & +s \\
\hline
 & & & & & \\
\hline
\end{array}$$

---

* Since the ſum of all the roots, in any equation, are equal to the co-efficient of the ſecond term, it follows that, when the ſecond term is wanting, the equation has both affirmative and negative roots, and that the ſum of the affirmative roots is equal to the ſum of the negative ones.

Thus in the cubic equation $x^3-7x=6$, the three roots are $+3$, -2, and -1, where it is evident that $3=2+1$.

PROBLEM III.

To find whether ſome or all the roots of an equation be rational; and, if ſo, what they are.

RULE *.

1. Find all the diviſors of the laſt term, and ſubſtitute them one by one for the unknown quantity.

2. Then, if the poſitive or negative terms deſtroy each other, the diviſor, ſo ſubſtituted, will be one of the roots of the equation.

3. But, if none of the diviſors ſucceed, the roots are, for the general part, either irrational or impoſſible.

Note, When the diviſors of the laſt term are too numerous, they may be diminiſhed by changing the equation into another, whoſe roots are augmented or decreaſed by an unit, or ſome other known quantity.

EXAMPLES.

1. Let $x^3-4x^2-7x+10=0$ be the equation propoſed.

Then the diviſors of (10), *the laſt term, will be* $+1$, -1, $+2$, -2, $+5$, -5, $+10$, -10.

---

[*] Since the laſt term, in any equation, is always equal to a multiple of all the roots in that equation, thoſe roots muſt, therefore, neceſſarily be found in the number of its diviſors.

But this, it is evident, can hold only when the roots are commenſurate, or whole numbers.

And theſe being ſubſtituted ſucceſſively inſtead of x, *will give*

$$\begin{array}{r}
1-4-7+10=0\\
-1-4+7+10=12\\
8-16-14+10=-12\\
-8-16+14+10=0\\
125-100-35+10=0
\end{array}$$

Therefore $+1$, -2 *and* $+5$ *are the three roots of the equation required.*

2. Let $y^4-4y^3-8y+32=0$, be the equation propoſed.

+ 1. *Change it into another, the number of whoſe diviſors ſhall be leſs; thus:*

Suppoſe $y=x+1$;

Then
$$\begin{array}{rl}
y^4= & x^4+4x^3+6x^2+4x+1\\
-4y^3= & -4x^3-12x^2-12x-4\\
-8y= & -8x-8\\
+32= & +32\\
\hline
 & x^4-6x^2-16x+21=0=
\end{array}$$ *new equation* *.

2, *The diviſors of the laſt term* (21) *of this new equation are*

$1, -1, +3, -3, +7, -7, +21, -21.$

And if theſe be ſubſtituted ſucceſſively inſtead of x, *we ſhall have*

$$\begin{array}{r}
1-6-16+21=0\\
1-6+16+21=32\\
81-54-48+21=0\\
81-54+48+21=96
\end{array}$$

&c. where none of the others ſucceed.

So that 1 *and* 3 *are the only rational roots, the other two being impoſſible.*

---

* *Note.* The diviſor of this new equation may be diminiſhed in the ſame manner as before.

+ The last term of ye equatn is = to ye prodt of all ye roots (v. 103) now if ye roots of an equatn be diminished, consq ye last term will be less complicated.

3. Let $x^3+3ax^2-4a^2x-12a^3=0$, be the equation propoſed.

Here the numeral diviſors of the laſt term ($12a^3$) *are* 1, -1, $+2$, -2, $+3$, -3, $+4$, -4, $+6$, -6, $+12$, -12.

And by ſubſtituting theſe ſucceſſively inſtead of x, *we ſhall have*

$$\begin{aligned}1+3-4-12&=-12\\-1+3+4-12&=-6\\8+12-8-12&=0\\-8+12+8-12&=0\\27+27-12-12&=30\\-27+27+12-12&=0\end{aligned}$$

Therefore the three roots are $2a$, $-2a$ *and* $-3a$.

PROBLEM IV.

To diſcover the roots of equations by SIR ISAAC NEWTON'*s method of diviſors.*

RULE.

1. Inſtead of the unknown quantity, ſubſtitute ſucceſſively three, or more, terms of the arithmetical progreſſion 2, 1, 0, -1, -2.

2. Collect all the terms of the equation into one ſum, and place them, together with their diviſors, in a perpendicular line, right againſt the correſponding terms of the progreſſion 2, 1, 0, -1, -2.

3, Seek amongſt the diviſors for an arithmetical progreſſion, whoſe terms correſpond with the order of the terms 2, 1, 0, -1, -2, and whoſe common difference is either an unit, or ſome diviſor of the co-efficient of the higheſt power of the unknown quantity in the given equation.

4. Divide that term of the progreſſion, thus found, which ſtands againſt the term 0 in the firſt progreſſion, by the ratio or common difference.

5. To the quotient, laſt found, prefix the ſign + or —, according as the progreſſion is increaſing or decreaſing, and this number being ſubſtituted for the unknown quantity, will give one of the roots of the equation.

EXAMPLES.

1. Let $x^3-x^2-10x+6=0$, be the equation propoſed.

Then, by ſubſtituting ſucceſſively the terms of the progreſſion 2, 1, 0, —1, *inſtead of* x, *the work will ſtand as follows:*

1ſt prog.	*reſults.*	*diviſors.*	*2d prog.*
2	—10	1 . 2 . 5 . 10	5
1	— 4	1 . 2 . 4	4
0	+ 6	1 . 2 . 3 . 6	3
—1	+14	1 . 2 . 7 . 14	2

And —3, *the term ſtanding againſt* 0, *being ſubſtituted for* x, *gives* $-27-9+30+6=0$; *and therefore* —3 *is a root of the equation.*

2. Let $2x^3-5x^2+4x-10=0$, be the equation propoſed.

Then, by ſubſtituting ſucceſſively the terms of the progreſſion 2, 1, 0, —1, —2, *inſtead of* x, *the work will ſtand as follows:*

1ſt prog.	*reſults.*	*diviſors.*	*2d prog.*
2	-6	1.2.3.6	1
1	-9	1.3.9	3
0	-10	1.2.5.10	5
-1	-21	1.3.7.21	7
-2	-54	1.2.3.6.9	9

Here 5, the term ſtanding againſt 0, being divided by 2, the common difference, gives $2\frac{1}{2}$; and this being ſubſtituted for x, gives $31\frac{1}{4}-31\frac{1}{4}+10-10=0$; and therefore $2\frac{1}{2}$ is a root of the equation.

3. Let $x^4+x^3-29x^2-9x+180=0$, be the equation propoſed *.

Then, by ſubſtituting as before, the work will ſtand as follows:

1ſt pro.	*reſults.*	*diviſors.*	*progreſſions.*			
2	70	1.2.5.7.10.14 &c.	1	2	5	7
1	144	1.2.3.4.6.8 &c.	2	3	4	6
0	180	1.2.3.4.5.6 &c.	3	4	3	5
-1	160	1.2.4.5.8.10 &c.	4	5	2	4
-2	90	1.2.3.5.6.9 &c.	5	6	1	3

So that here are four progreſſions, and the numbers 3, 4, -3, and -5, being each ſubſtituted for x, make the whole equation vaniſh, and are therefore the roots required.

4. Given $x^3-3x^2-46x-72=0$, to find the values of x, by the method of diviſors. *Anſ.* $+9$, -2 *and* -4.

5. Given $x^4-4x^3-19x^2+46x+120=0$, to find the values of x, by the method of diviſors.

Anſ. $+5, +4, -3, -2$.

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* Several other rules for diſcovering the roots of equations may be found in Newton's and Maclaurin's Algebra.

PROBLEM V.

To find the roots of cubic equations, according to the method of CARDAN.

RULE *.

1. Take away the ſecond term of the equation, by problem 2d, and it will be reduced to this form $x^3 \pm ax = \pm b$.

2. Subſtitute the values of a and b, with their proper ſigns, in the following expreſſion, and it will give the root required. Thus:

$$x = \sqrt[3]{\tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)}} - \frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)}}} =$$

root required.

---

[*] The rule, from whence this method is derived, is $x = \sqrt[3]{\tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)}} + \sqrt[3]{\tfrac{1}{2}b - \sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)}}$; and the inveſtigation of it is as follows:

Let the equation, whoſe root is required, be $x^3 + ax = b$.

And aſſume $y + z = x$, and $3yz = -a$. $yz = -\frac{a}{3}$; $z = -\frac{a}{3y}$

Then, by ſubſtituting theſe values in the given equation, we ſhall have, $y^3 + 3y^2z + 3yz^2 + z^3 + a \times (y+z) = y^3 + z^3 + 3yz \times (y+z) + a \times (y+z) = y^3 + z^3 - a \times (y+z) + a \times (y+z) = y^3 + z^3 = b$. $y^6 + 2y^3z^3 + z^6 = b^2 =$ Square.

$4y^3z^3 = -\frac{4a^3}{27}$; $+$

And if, from the ſquare of this laſt equation, there be taken 4 times the cube of the equation $yz = -\tfrac{1}{3}a$, we ſhall have $y^6 - 2y^3z^3 + z^6 = b^2 + \tfrac{4}{27}a^3$, or $y^3 - z^3 = \sqrt{(b^2 + \tfrac{4}{27}a^3)}$.

But the ſum of this equation and $y^3 + z^3 = b$, is $2y^3 = b + \sqrt{(b^2 + \tfrac{4}{27}a^3)}$; and their difference is $2z^3 = b - \sqrt{(b^2 + \tfrac{4}{27}a^3)}$; whence y is found $= \sqrt[3]{\tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)}}$, and $z = \sqrt[3]{\tfrac{1}{2}b - \sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)}}$.

$$2y^3 = b + \sqrt{b^2 + \frac{4a^3}{27}} = \text{Sum}$$

$$y^3 = \frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{\cancel{4}a^3}{\cancel{4}27}}$$

$$2z^3 = b - \sqrt{b^2 + \frac{4a^3}{27}} = \text{Difference}$$

$$z^3 = \frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$$

Note *. When a is negative, and $\frac{1}{27}a^3$ is greater than $\frac{1}{4}b^2$, the ſolution, by this rule, cannot be generally obtained.

EXAMPLES.

1. Let $y^3+3y^2+9y=13$, be the equation propoſed; it is required to find the value of y.

1. *In order to deſtroy the ſecond term, let* $y=x-1$; *then*

$$\begin{aligned} y^3 &= x^3-3x^2+3x-1 \\ 3y^2 &= +3x^2-6x+3 \\ 9y &= +9x-9 \\ \hline & x^3+6x-7=13 \\ & \text{or } x^3+6x=20. \end{aligned}$$

And from hence it appears, that $y+z$, or its equal x, is $= \sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}+\sqrt[3]{\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}$, which is the theorem.

Or, ſince z is $=-\frac{a}{3y}$, it will be $y+z=y-\frac{a}{3y}=x=$
$\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}}$, *the ſame as the rule.*

This method of ſolving cubic equations is uſually aſcribed to *Cardan*; but the invention is not his.—The authors of it were *Scipio Ferreus* and *Nicholas Tartalea*, who diſcovered it about the ſame time, independently of each other, as is proved by *M. de Montucla* in his *Hiſtoire des Mathématiques*.

* This is called the irreducible caſe; and, notwithſtanding ſeveral of the moſt eminent mathematicians in Europe have attempted the ſolution of it, no general rule has yet been diſcovered.

The uſual method is by a table of ſines, or by throwing the expreſſion into an infinite ſeries, and finding the ſum of a certain number of terms, according to the degree of exactneſs required.

2. *For a put* 6, *and for b* 20, *and we ſhall have*

$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}}=$$

$$\sqrt[3]{10+\sqrt{(100+8)}}-\frac{2}{\sqrt[3]{10+\sqrt{(100+8)}}}=$$

$$\sqrt[3]{10+10.3923}-\frac{2}{\sqrt[3]{10+10.3923}}=\sqrt[3]{20.3923}-$$

$$\frac{2}{\sqrt[3]{20.3923}}=2.732-\frac{2}{2.732}=2.732-.732=2;$$ *that is* $x=2$, *and conſequently* $y=1=$ *root required.*

2. Given $x^3-6x=-9$, to find the value of x.

Here $a=-6$, *and* $b=-9$; *and therefore we ſhall have*

$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}}=$$

$$\sqrt[3]{-4\tfrac{1}{2}+\sqrt{(20\tfrac{1}{4}-8)}}-\frac{-2}{\sqrt[3]{-4\tfrac{1}{2}+\sqrt{(20\tfrac{1}{4}-8)}}}=$$

$$\sqrt[3]{-4\tfrac{1}{2}+3\tfrac{1}{2}}-\frac{-2}{\sqrt[3]{-4\tfrac{1}{2}+3\tfrac{1}{2}}}=\sqrt[3]{-1}-\frac{-2}{\sqrt[3]{-1}}=-1$$

$$-\frac{-2}{-1}=-1-2=-3;$$ *that is* $x=-3=$ *root required.*

EXAMPLES FOR PRACTICE.

1. Given $x^3-6x^2+10x=8$ to find x. *Anſ.* $x=4$.
2. Given $y^3+30y=117$, to determine y. *Anſ.* $y=3$.
3. Given $y^3-36y=91$, to determine y. *Anſ.* $y=7$.
4. Given $y^3-3y=18$, to determine y. *Anſ.* $y=3$.
5. Given $y^3+24y=250$, required y. *Anſ.* $y=5.05$.
6. Given $y^6-3y^4-2y^2-8=0$, to find y. *Anſ.* $y=2$.

PROBLEM VI.

To find the roots of biquadratic equations, according to the method of DES CARTES.

RULE *.

1. Take away the ſecond term of the equation by problem 2, and it will be reduced to the form $x^4+qx^2+rx+s=0$.

2. From the cubic equation $y^3+2qy^2+(q^2-4s)y-r^2=0$ take the ſecond term, and find the value of y by the laſt problem.

3. Let e be aſſumed $=\sqrt{y}$, $f=\frac{1}{2}q+\frac{1}{2}e^2-\frac{r}{2e}$, and $g=\frac{1}{2}q+\frac{1}{2}e^2+\frac{r}{2e}$.

4. Find the roots of the two quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0$, and they will be the four roots of the biquadratic required.

---

[*] *Inveſtigation of the rule.* Let the given equation $x^4+qx^2+rx+s=0$, be equal to the product of the two quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0=x^4+(f+g-e^2)x^2+(eg-ef)x+fg=0$.

Then, by equating the homologous terms, we ſhall have $f+g-e^2=q$, $eg-ef=r$, and $fg=s$; and therefore $f=$ * $\frac{1}{2}q+\frac{1}{2}e^2-\frac{r}{2e}$, $g=\frac{1}{2}q+\frac{1}{2}e^2+\frac{r}{2e}$, and $s=f\times g=\frac{1}{4}q^2+\frac{1}{2}qe^2+\frac{1}{4}e^4-\frac{r^2}{4e^2}$.

And from this laſt equation we ſhall have $e^6+2qe^4+(q^2-4s)\times e^2=r^2=$ to a cubic equation, in which the value of e may be found, as in the laſt problem.

* By extract.g ye balue of f, in both equations
$f=q-g+e^2$; & $f=-\frac{r}{e}+g$; then by tak.g ye equali
$q-g+e^2=-\frac{r}{e}+g$; $2g=q+e^2+\frac{r}{e}$; $g=\frac{q}{2}+\frac{e^2}{2}+\frac{r}{2e}$
by substit.g ye value of g, in f, $f=q-\frac{q}{2}-\frac{e^2}{2}-\frac{r}{2e}+e^2$ *
by red.g ye last equat.n into their simple terms, we shall h
$f=\frac{q}{2}+\frac{e^2}{2}-\frac{r}{2e}$; & $g=\frac{q}{2}+\frac{e^2}{2}+\frac{r}{2e}=s$. & by ×.g them

EXAMPLES.

1. Let $z^4-4z^3-8z+32=0$, be the equation propoſed, in which it is required to find the value of z.

1. *To take away the ſecond term*, *let* $x+1=z$; *then*

$$\begin{aligned}
z^4 &= x^4+4x^3+6x^2+4x+1\\
-4z^3 &= -4x^3-12x^2-12x-4\\
-8z &= -8x-8\\
+32 &= +32\\
\hline
& x^4-6x^2-16x+21=0, \text{ or}\\
& y^3-12y^2-48y-256=0, \text{ for the cubic equation.}
\end{aligned}$$

2. *To take away the ſecond term from this equation*, *let* $p+4=y$; *then*

$$\begin{aligned}
y^3 &= p^3+12p^2+48p+64\\
-12y^2 &= -12p^2-96p-192\\
-41y &= -48p-192\\
-256 &= -256\\
\hline
& p^3-96p=576
\end{aligned}$$

But $f\left(=\frac{1}{2}q+\frac{1}{2}e^2-\frac{r}{2e}\right)$ and $g\left(=\frac{1}{2}q+\frac{1}{2}e^2+\frac{r}{2e}\right)$ are alſo known; and therefore the roots of the quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0$, may be determined, and are the four roots of the biquadratic equation required. Q. E. I.

Note, The co-efficient of x is put equal to e, in both the equations, becauſe, when the ſecond term is wanting, the ſum of the poſitive roots is always equal to the ſum of the negative ones, with a contrary ſign.

This rule has ſometimes been aſcribed to *Des Cartes*, and ſometimes to *Bombelli*, an Italian; but the original inventor of it was *Louis Ferrari*.

together, we shall have for their product =

$\frac{q^2}{4}+\frac{qe^2}{2}+\frac{e^4}{4}-\frac{r^2}{4e^2}=s$; & by X^g. these by $4e^2$; in order to cancel ye unknown quanty in ye denom.

$q^2e^2+2qe^4+e^6-r^2=4e^2s=e^6+2qe^4+(q^2-4s)e^2=r^2$

3. *To find the value of p, by* CARDAN'S *rule for cubic equations.*

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}}=$$

$$\sqrt[3]{288+\sqrt{(288^2-32^3)}}-\frac{-32}{\sqrt[3]{288+\sqrt{(288^2-32^3)}}}=$$

$12=p$; *and therefore* $y=16$, *or* $\sqrt{y}=4$, $f=\frac{-6}{2}+\frac{16}{2}+\frac{16}{8}=7$, *and* $g=\frac{-6}{2}+\frac{16}{2}-\frac{16}{8}=3$. *Whence* $e=4$, $f=7$ and $g=3$.

In the method of *Des Cartes*, above explained, all biquadratic equations are ſuppoſed to be generated from the multiplication of two quadratic ones: but, according to the following way, which is taken from *Simpſon's Algebra*, every ſuch equation is conceived to ariſe by taking the difference of two complete ſquares.

Here, the general equation $x^4+ax^3+bx^2+cx+d=0$ being propoſed, we are to aſſume $(x^2+\frac{1}{2}ax+A)^2-(Bx+C)^2=x^4+ax^3+bx^2+cx+d$; in which A, B, and C, repreſent unknown quantities to be determined.

Then, $x^2+\frac{1}{2}ax+A$, and $Bx+C$ being actually involved, will give $\left.\begin{matrix}x^4+ax^3+2Ax^2\\ +\frac{1}{4}a^2x^2+aAx\ +A^2\\ -B^2x^2-2BCx-C^2\end{matrix}\right\}=x^4+ax^3+bx^2$ = ye given $+cx+d$: from whence, by equating the homologous terms, we ſhall have

1. $2A+\frac{1}{4}a^2-B^2=b$, or $2A+\frac{1}{4}a^2-b=B^2$;
2. $aA-2BC\qquad=c$, or $aA-c\qquad=2BC$; +
3. $A^2-C^2\qquad=d$, or $A^2-d\qquad=C^2$.

Let now the firſt and laſt of theſe equations be multiplied together, and the product will, evidently, be equal to $\frac{1}{4}$ of the ſquare of the ſecond; that is $2A^3+(\frac{1}{4}a^2-b)\times A^2-2dA-(\frac{1}{4}a^2-b)\times d(=B^2C^2)=\frac{1}{4}\times(a^2A^2-2acA+c^2)$.

M

By raisg ye equatn to a Square

$\frac{a^2A^2-2acA+c^2}{4}=\frac{4B^2C^2}{4}=\frac{1}{4}(a^2A^2-2acA+c^2=B^2C^2$

Then $2A^3+(\frac{a^2}{4}-b)A^2-2dA-(\frac{1}{4}a^2-b)d=\frac{1}{4}(a^2A^2-2acA+$

Then by developg the equations, $2A^3+\frac{a^2}{4}A^2-bA^2-2dA-\frac{a^2}{4}d+bd=\frac{a^2}{4}A^2+\frac{acA}{2}+\frac{c^2}{4}$; Now by rangg ym accg to ye highest power of ye unknown quantity; = to

4. To find the roots of the two quadratic equations $x^2+ex+f=0$, *and* $x^2-ex+g=0$.

$$x^2+ex+f=x^2+4x+7=0$$
$$x^2-ex+g=x^2-4x+3=0$$

In the firſt of theſe $x=-2+\sqrt{-3}$, *or* $-2-\sqrt{-3}$.

And in the ſecond $x=3$, *and* 1.

Therefore 3, 1, $-2+\sqrt{-3}$, *and* $-2-\sqrt{-3}$ *are the four roots of the equation* $x^4-6x^2-16x+21=0$.

And if unity be added to each of them we ſhall have 4, 2, $-1+\sqrt{-3}$, *and* $-1-\sqrt{-3}$ *for the roots of* $z^4-4z^3-8z+32=0$, *the equation propoſed; the two laſt of which are impoſſible.*

2. Given $x^4+2x^3-7x^2-8x+12=0$, to find the values of x. *Anſ.* $x=1$, 2, -3 *and* -2.

Whence, by denoting the given quantities $\frac{1}{4}ac-d$, and $\frac{1}{4}c^2+d\times(\frac{1}{4}a^2-b)$ by k and l, reſpectively, there will ariſe this cubic equation, $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=0$; by means of which the value of A may be determined; and therefore, from the preceding equations, both B and C will alſo be known; B being found from thence $=\sqrt{(2A+\frac{1}{4}a^2-b)}$, and $C=\frac{aA-c}{2B}$.

The ſeveral values of A, B, and C, being thus found, that of x will alſo be readily obtained: for $(x^2+\frac{1}{2}ax+A)^2-(Bx+C)^2$ being univerſally, in all circumſtances of x, equal to $x^4+ax^3+bx^2+cx+d$, it is evident, that, when the value of x is taken ſuch that the latter of theſe expreſſions becomes equal to nothing, the former muſt likewiſe be equal to nothing; and conſequently $(x^2+\frac{1}{2}ax+A)^2=(Bx+C)^2$.

And therefore, by extracting the ſquare root of both ſides of * the equation, we ſhall have $x^2+\frac{1}{2}ax+A=\pm Bx\pm C$; or $x=\pm\frac{1}{2}B-\frac{1}{4}a\pm(\overline{(\frac{1}{4}a\mp\frac{1}{2}B)^2}\pm C-A)^{\frac{1}{2}}=\pm\frac{1}{2}B-\frac{1}{4}a\pm(\frac{1}{16}a^2\mp\frac{1}{4}aB+\frac{1}{4}B^2\pm C-A)^{\frac{1}{2}}$, which exhibits all the different roots of the given equation, according to the variation of the ſigns.

$$2A^3-bA^2-2d.A+\frac{a\dot{c}A}{2}-\frac{a^2d}{4}+bd-\frac{\dot{c}^2}{4}=$$

$$2A^3-bA^2-(2d-\frac{a\dot{c}}{2})A-(\frac{a^2}{4}-b)d-\frac{\dot{c}^2}{4};$$ ÷ it by 2

$$A^3-\frac{b}{2}A^2-(d-\frac{ac}{4})A-\frac{(\frac{a^2}{4}-b)d-\frac{c^2}{4}}{2}$$

$$\frac{a\dot{c}}{4}-d=k;\ \frac{\dot{c}^2}{4}+d(\frac{a^2}{4}-b)=L;\ A^3-\frac{b}{2}A^2+kA-\frac{L}{2}$$

$=0$ * By arrang^g & complet^g y^e square of y^e Quadratic

3. Given $x^4-25x^2+60x-36=0$, to find the values of x. *Anſ.* 1, 2, 3 *and* −6.

4. Given $y^4-8y^3+14y^2+4y-8=0$, to find the values of y.

Anſ. $y=3+\sqrt{5}$, $3-\sqrt{5}$, $1+\sqrt{3}$, *and* $1-\sqrt{3}$.

5. Given $x^4+12x-17=0$, to find the values of x.
Anſ. $x=\frac{1}{2}\sqrt{2}\pm\sqrt{(-3\sqrt{2}-\frac{1}{2})}$ *and alſo* $-\frac{1}{2}\sqrt{2}\pm\sqrt{(3\sqrt{2}-\frac{1}{2})}$.

6. Given $x^4-4x^3-3x^2-4x+1=0$, to find the values of x. *Anſ.* $x=\frac{-1\pm\sqrt{-3}}{2}$ *and* $\frac{5\pm\sqrt{21}}{2}$.

7. Given $x^4-10x^3+35x^2-50x+24=0$, to find the values of x. *Anſ.* 1, 2, 3 *and* 4.

This method will be found to have many advantages over that given above. In the firſt place, there is no neceſſity for being at the trouble of exterminating the ſecond term of the equation, in order to prepare it for a ſolution: ſecondly, the equation $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=0$, here brought out, is of a more ſimple form than that derived from the former method: and thirdly, the value of A in this equation will always be *commenſurate* and *rational*; not only when all the roots of the given equation are *commenſurate*, but alſo when they are *irrational*, and even *impoſſible*.

1.

Example. *Let there be given* $x^4+12x-17=0$, *to find the values of* x.

Here, by comparing this with the general equation $x^4+ax^3+bx^2+cx+d=0$, we ſhall have $a=0$, $b=0$, $c=12$, and $d=-17$; and therefore $k=\frac{1}{4}ac-d=17$, $l=\frac{1}{4}c^2+d\times(\frac{1}{4}a^2-b)=36$, and $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=A^3+17A-18=0$.

* And, from this equation, A will be found equal to 1; and therefore $B=(2A+\frac{1}{4}a^2-b)^{\frac{1}{2}}=\sqrt{2}$, $C=\frac{aA-c}{2B}=\frac{-12\times\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\frac{-12\sqrt{2}}{4}$ $=-3\sqrt{2}$, and $x=\pm\frac{1}{2}\sqrt{2}\pm(\frac{1}{2}\mp3\sqrt{2}-1)^{\frac{1}{2}}=\pm\frac{1}{2}\sqrt{2}\mp(\pm3\sqrt{2}-\frac{1}{2})^{\frac{1}{2}}$. or $x=\pm\frac{1}{2}\sqrt{2}\pm(\mp3\sqrt{2}-\frac{1}{2})^{\frac{1}{2}}$ Vid. P. 125

$x^2+(\frac{a}{2}\mp B)x=-A\pm C$; $x^2+(\frac{a}{2}\mp B)x+(\frac{a}{4}\mp\frac{B}{2})^2=(\frac{a}{4}\mp\frac{B}{2})^2-A\pm C$; Then $x+(\frac{a}{4}\mp\frac{B}{2})=\sqrt{(\frac{a}{4}\mp\frac{B}{2})^2-A\pm C}$;

$x=\pm\frac{B}{2}-\frac{a}{4}\pm\sqrt{(\frac{a}{4}\mp\frac{B}{2})^2-A\pm C}$; By expos.^{n} of y quant.^{s}

$x=\pm\frac{B}{2}-\frac{a}{4}\pm(\frac{a^2}{16}-\frac{a}{4}B+\frac{B^2}{4}\pm C-A)^{\frac{1}{2}}$;

* $A^3+17A=18$; By Cardan's Rule, $A=\sqrt[3]{9+\sqrt{81+\frac{4913}{27}}}$

8. Given $x^4-6x^3-58x^2-114x-11=0$, to find the values of x.

Anſ. $x=\pm\frac{5}{2}\sqrt{3}+\frac{3}{2}\pm\sqrt{(17\pm\frac{21}{2}\sqrt{3})}$.

9. Given $x^4-3x^2-4x-3=0$, to find the values of x.

Anſ. $x=\frac{1\pm\sqrt{3}}{2}$ *and* $\frac{-1\pm\sqrt{-3}}{2}$.

PROBLEM VII.

To find the roots of equations in general, by the methods of approximation and converging ſeries.

RULE *.

1. Find, by trial, a number nearly equal to the root required.

In ſome particular caſes of this rule, the roots may be found by means of quadratics only.

Several other methods of ſolving biquadratic equations have been invented by different authors; but one of the moſt ingenious is that given by *M. Euler*, in page 664 of his *Elémens D' Algebre.*

Equations of five or more dimenſions may be reduced to thoſe of an inferior degree; but the proceſs will be exceedingly tedious, as no general rule can be given for reſolving them.

* The rules hitherto given, for finding the roots of equations, are either very troubleſome and laborious, or elſe confined to particular caſes; but this method, by converging ſeries, is univerſal, extending to all kinds of equations whatever; and, though not accurately true, gives the value ſought to any aſſigned degree of exactneſs.

The method of obtaining the roots of equations, by approximation, was firſt made uſe of by *Vieta*.

$-\sqrt[3]{85}\,\frac{57}{3}$ &c. $A=\sqrt[3]{9+\frac{1}{3}\sqrt{\frac{7100}{3}}}$; after ÷ ye square root by 3,

$A=\sqrt[3]{9+\frac{1}{3}(48,648398}$, after tak.g ye 3.d part

$A=\sqrt[3]{25,216132}$; $A=2,93-\frac{5,66}{2,93}$; got by ÷ 17 [illegible]

$A=2,93-1,93=1,00$; obt.d by ÷ 5,66, by 2,93.

2. Call the number, thus found, r, and let z be put equal to the difference between r and the true root x.

3. Inſtead of x, in the given equation, ſubſtitute its equal $r \pm z$, and there will ariſe a new equation, affected only with z and known quantities.

4. Reject all thoſe quantities in which there are two or more dimenſions of z, and the value of z will be found by means of a ſimple equation.

5. Add the value of z, thus found, to the value of r, and it will give the root required *nearly*.

6. If this root is not ſufficiently near the truth, repeat the operation, by ſubſtituting it inſtead of r, in the equation exhibiting the value of z, and it will give a *ſecond correction* for the root required.

EXAMPLES.

1. Given $x^2-5x-31=0$, to find the value of x by approximation.

The root, found by trial, is nearly equal to 8;
Let, therefore, $8=r$, *and* $r+z=x$; *then*

$$\begin{aligned} x^2 &= r^2+2rz+z^2 \\ -5x &= -5r-5z \\ -31 &= -31 \end{aligned}$$

$$r^2+2rz-5r-5z-31=0;$$

or $z=\dfrac{31+5r-r^2}{2r-5}=\dfrac{31+40-64}{16-5}=\dfrac{7}{11}=.6$, *and* $x=$ 8.6 *nearly*.

And, again, if 8.6 *be ſubſtituted in the place of* r *in the laſt equation, we ſhall have*

$$z=\frac{31+5r-r^2}{2r-5}=\frac{31+43-73.96}{17.2-5}=\frac{.04}{12.2}=.0032$$

and $x=8.6+.0032=8.6032$ *nearly*.

$x=\pm\frac{1}{2}\sqrt{2}\pm(\mp\frac{1}{2}\sqrt{2})^2\mp 3\sqrt{2}-1)^{\frac{1}{2}}$; Expos. ye 3d intermed. No

$x=\pm\frac{1}{2}\sqrt{2}\pm(\frac{1}{4}\times 2 \text{ or } \frac{1}{2})\mp 3\sqrt{2}-1)^{\frac{1}{2}}$

$x=\pm\frac{1}{2}\sqrt{2}\pm(\frac{1}{2}\mp 3\sqrt{2}-\frac{2}{2})^{\frac{1}{2}}$; $x=\pm\frac{1}{2}\sqrt{2}\pm(\mp 3\sqrt{2}-\frac{1}{2})^{\frac{1}{2}}$

or $x=\pm\frac{1}{2}\sqrt{2}\mp(\pm 3\sqrt{2}-\frac{1}{2})^{\frac{1}{2}}$ In either cases it gives ye same result.

And, if this value be again ſubſtituted for r, it will give $z=.000077808$, *and* $x=8.603277808$; *and ſo on to any degree of exactneſs required.*

2. Given $x^3+x^2+x=90$, to find the value of x by approximation.

The root, found by trial, is nearly equal to 4;
Let, therefore, $4=r$, *and* $r+z=x$, *then,*

$$\begin{aligned} x^3 &= r^3+3r^2z+3rz^2+z^3 \\ x^2 &= r^2+2rz+z^2 \\ x &= r+z \\ \hline r^3+3r^2z&+r^2+2rz+r+z=90; \end{aligned}$$

or, $z=\dfrac{90-r^3-r^2-r}{3r^2+2r+1}=\dfrac{90-64-16-4}{48+8+1}=\dfrac{6}{57}=.10,$
and $x=4.1$ *nearly*

And agin, if 4.1 *be ſubſtituted in the place of r, in the laſt equation, we ſhall have*

$$\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-68.921-16.81-4.1}{50.43+8.2+1}=.00283,$$

and $x=4.1+.00283=4.10283$ *nearly; and ſo on to any degree of exactneſs required.*

+ 3. Given $x^2+20x=100$, to find the value of x by approximation. *Anſ.* $x=4.1421357$.

4. Given $x^3+10x^2+5x=2600$, to find the value of x by approximation. *Anſ.* 11.00679.

5. Given $x^3+2x^2-23x-70=0$, to find the value of x. *Anſ.* $x=5.1345$.

6. Given $x^3-15x^2+63x-50=0$, to find the value of x. *Anſ.* $x=1.028039$.

7. Given $x^4-3x^2-75x=10000$, to find the value of x. *Anſ.* $x=10.2615$.

8. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x. *Anſ.* $x=8.4144$.

+ x = 4,14. 1st. degree, x = Ansr. 2d deg of approx.
* x = 11,006; 1st. degree, x = Ansr. 2d. deg of approx.
5 x = 5,13. 1st. degree; x = ansr. 2d. deg of approx.

RULE II.

1. Aſſume the general equation $az+bz^2+cz^3+dz^4$, &c. $=p$; where z is the converging quantity, and a, b, c, d, &c. co-efficients whoſe values are known.

2. Then will $\frac{ap}{a^2+bp}$ be an approximation of the firſt degree.

3. And, if s be put $=\frac{b}{a}-\frac{c}{b}$ we ſhall have $\frac{(a+sp)\times p}{a^2+(b+as)\times p}$ for an approximation of the ſecond degree.

4. And, in like manner, if w be put $=\frac{2b}{a}+\frac{ad-bc}{b^2-ac}$, then will $\frac{ap\times(a+wp)}{a\times(a^2+(b+aw)\times p)+(w-s)\times p^2}$ be an approximation of the third degree, &c.

EXAMPLES.

1. Given $x^2+20x=100$, to find the value of x.

The root, found by trial, is nearly equal to 4;

Let therefore $4+z=x$, *and, by ſubſtitution, the equation will become* $28z+z^2=4$.

Whence, from the rule, $a=28$, $b=1$, $c=0$, *&c. and* $p=4$.

Therefore $z=\frac{ap}{a^2+bp}=\frac{112}{788}=\frac{28}{197}=.14213$ *for the firſt approximation.*

And, ſince $s=\frac{b}{a}-\frac{c}{b}=\frac{1}{28}$, *we ſhall have*

$$z=\frac{(a+sp)\times p}{a^2+(b+as)\times p}=\frac{(28+\frac{1}{7})\times 4}{28\times 28+(1+1)\times 4}=\frac{28+\frac{1}{7}}{28^2\times 7+2}$$

$=\frac{197}{1386}=.14213564$ *for the ſecond approximation.*

And, in like manner, ſince $w=\frac{2b}{a}+\frac{ad-bc}{b^2-ac}=\frac{1}{14}$, *it will be* $z=\frac{ap\times(a+wp)}{a\times(a^2+(b+aw)\times p)+(w-s)\times p^2}=$

$$\frac{28\times 4\times(28+\frac{2}{7})}{28\times(784+12)+\frac{4}{7}}=\frac{28\times(28+\frac{2}{7})}{7\times 796+\frac{1}{7}}=\frac{28\times 198}{49\times 796+1}=$$

$\frac{5544}{39005}=.1421356236$; *or* $x=4.1421356236$ *for the root required, extremely near.*

2. Given $x^2+4x-10=100$, to find the value of x. *Anſ.* 8.677078.

3. Given $x^3-17x^2+54x=350$, to find the value of x. *Anſ.* 14.95407.

4. Given $x^3-2x-5=0$, to find the value of x. *Anſ.* 2.094551.

5. Given $2y^4-16y^3+40y^2-30y=-1$, to find the value of y. *Anſ.* $y=1.284724$.

6. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x. *Anſ.* $x=8.414455$.

In the ſame manner the roots of other equations may be approximated; but, to avoid trouble in preparing the equation for a ſolution, all ſuch powers of the converging quantity z, as would riſe higher than the degree, or order of the approximation you intend to work by, may be every where neglected.

PROBLEM VIII.

To extract the root of any pure power in numbers.

RULE *.

1. Let $m=$ the number whoſe root is required; $r=$ neareſt root which can be found by trial; and $n=$ to the index.

2. Then, by putting $v=\frac{nr^n}{m-r^n}$, we ſhall have $x=r+\frac{r\times(6v+n+1)}{v\times(6v+4n-2)}=$ root, *nearly*; or $x=r+\frac{r\times(2v+n)}{v\times(2v+2n-1)+\frac{1}{6}\times(n-1)\times(2n-1)}$ *extremely near.*

EXAMPLES.

1. Given $x^2=2$; or, which is the ſame thing, let the ſquare root of 2 be found.

Suppoſe the root, found by trial, to be 1.4; *then we ſhall have* $m=2$, $r=1.4$, $n=2$, *and* $v=\frac{2\times1.96}{2-1.96}$ $=98$.

---

[*] One of the moſt convenient rules for practice, which has yet been diſcovered, is the following: $(n+1)r^3+(n-1)m : (n+1)m+(n-1)r^3 :: r :$ the true root *nearly*.

Where it may be obſerved that $m=$ given number; $r=$ neareſt root, found by trial; and $n=$ index, as before.

And, therefore, $x = r + \frac{r\times(6v+\ n+1)}{v\times(6v+4n-2)} = 1.4 + \frac{1.4\times591}{98\times594} = 1.4 + \frac{197}{70\times198} = 1.4 + \frac{197}{13860} = 1.41421356$ = *root required nearly.*

And if the ſecond approximation be uſed, the root will be found $= 1.414213562\dot{3}6$, *which is true to the laſt place of decimals.*

2. Given $x^3 = 500$; or let it be required to extract the cube root of 500.

Suppoſe the root, found by trial, to be 8; *then we ſhall have* $m = 500$, $r = 8$, $n = 3$, *and* $v = \frac{3\times512}{-12} = -128$;

And, therefore, $x = r + \frac{r\times(6v+\ n+1)}{v\times(6v+4n)-2} = 8 - .063 = 7.93$ *for the 1ſt approximation.*

Or $x = r + \frac{r\times(2v+n)}{(2v+2n-1)\times v + \frac{1}{6}\times(n-1)\times(2n-1)} = 8 - \frac{6072}{96389} = 7.937005259936$ *for the 2d approximation, which is true to the laſt place of decimals.*

3. Let it be required to find the cube root of 2.
Anſ. 1.259921.

4. Required the cube root of 117? *Anſ.* 4.89097.

5. What is the ſurſolid, or 5th root, of 125000?
Anſ. 10.456389.

6. It is required to find the 7th root of 100000.
Anſ. 5.1794746792.

7. It is required to find the 365th. root of 1.05.
Anſ. 1.00013366.

PROBLEM IX.

To find the root of an exponential equation.

RULE *.

1. Find, by trial, two numbers, as near the true root as poſſible, and ſubſtitute them in the given equation inſtead of the unknown quantity, marking the errors which ariſe from each of them.

2. Multiply the difference of the two numbers, found by trial, by the leaſt error, and divide the product by the difference of the errors, when they are alike, and by their ſum when they are unlike.

3. Add the quotient, laſt found, to the number belonging to the leaſt error, when that number is too little, and ſubtract it when too great, and the reſult will give the true root *nearly*.

4. Take this root and the neareſt of the former, and, by proceeding in like manner, a root will be had ſtill nearer than before; and ſo on to any degree of exactneſs required.

EXAMPLES.

1. Given $x^x = 100$, to find the value of x by approximation.

By the nature of logarithms $x \times$ *log.* $x =$ *log.* $100 = 2$.

And, ſince x *is found, by trial, to be greater than* 3 *and leſs than* 4;

Let, therefore, 3.5 *and* 3.6 *be the two ſuppoſed values of* x.

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* The rule for ſolving exponential equations was invented by *M. Jean Bernoulli*, and publiſhed in the *Leipſic* Acts, 1697.

Then the log. $x=$ *log.* $3.5=.5440680$; *and* $x\times$ *log.* $x=1.9042380$

2

$-.0957620=$ *1st error, too little*

And the log. $x=$ *log.* $3.6=.5563025$; *and* $x\times$ *log.* $x=2.0026890$

2

$.0026890=$ *2d error, too great.*

1st number 3.5 *1st error* $-.095762$
2d number 3.6 *2d error* $+.002689$

$0.1=$ *diff,* $.098451=$ *sum.*

$$\frac{0.1\times.002689}{.098451}=.00273=\text{correction,}$$

2d number 3.60000
correction $-.00273$

$3.59727=x=$ *root nearly.*

Again, suppose $x=3.597$; *then we shall have* *log.* $x=.5559404$, *and* $x\times$ *log.* $x=1.9997176$, *which subtracted from* 2 *gives* $.0002824$ *the 3d error, too little.*

2d number 3.600 *2d error* $+.0026890$
3d number 3.597 *3d error* $-.0002824$

$.003=$ *diff.* $.0029714=$ *sum.*

$$\frac{.003\times.0002824}{.0029714}=.000285\ \text{the correction.}$$

3d number 3.597000
correction $+0.000285$

$3.597285=x=$ *root required nearly.*

2. Given $x^x=123456789$ to determine the value of x. *Ans.* $x=8.6400268$.

OF
INDETERMINATE OR UNLIMITED PROBLEMS.

A problem is ſaid to be *indeterminate* or *unlimited,* when the equations, expreſſing the conditions of a queſtion, are leſs in number than the unknown quantities to be determined.

And though ſuch kind of problems are capable of innumerable anſwers, yet the reſults, in whole numbers, are generally limited to ſome determinate number, and may be obtained as follows.

PROBLEM I.

To find the values of x *and* y*, in the equation* $ax = by+c$; *where* a*,* b*, and* c*, are given numbers, which admit of no common diviſor.*

RULE*.

1. Let *wh.* ſtand for a *whole number,* and reduce the equation to $x = \frac{by+c}{a} = wh.$

2. Make $\frac{by+c}{a} =$ to $\frac{dy+f}{a}$, by throwing all whole numbers out of it, till d and f be each leſs than a.

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[*] This rule is founded on the following obvious principles:

That the ſum, difference, or product, of any two whole numbers, is a whole number.

And that if a number meaſures the whole of any number, and a part of it, it will alſo meaſure the remaining part.

3. Subtract $\frac{dy+f}{a}$, or ſome multiple of it, from $\frac{ay}{a}$, $\frac{2ay}{a}$, $\frac{3ay}{a}$, or any other multiple of y, that comes near the former, and the remainder will be a whole number.

4. Take this remainder, or any multiple of it, from ſome of the foregoing fractions, or from any whole number, which is nearly equal to it, and the remainder, in this caſe, will alſo be a whole number.

5. Proceed in the ſame manner with this laſt remainder; and ſo on till the co-efficient of y becomes equal to 1; or $\frac{y+g}{a}=wh.=p$.

6. Then will $y=ap-g$; where p may be any whole number whatever; and as the value of y is now known, that of x may alſo be found from the given equation.

EXAMPLES.

1. Given $19x=14y-11$, to find x and y in whole numbers.

Firſt, $x=\frac{14y-11}{19}=wh.$; *and* $\frac{19y}{19}=wh.$

Then, by ſubtraction, $\frac{19y}{19}-\frac{14y-11}{19}=\frac{5y+11}{19}=wh.$

Again, $\frac{5y+11}{19}\times 4=\frac{20y+44}{19}=\frac{20y+6}{19}+2=wh.$

and, by rejecting the 2, $\frac{20y+6}{19}=wh.$

Therefore $\frac{20y+6}{19}-\frac{19y}{19}=\frac{y+6}{19}=wh.=p.$

And $y=19p-6$; *where, if* p *be taken* $=1$, *for the leaſt affirmative value of* y, *we ſhall have* $y=13$, *and* $x=9$, *the anſwer.*

2. Given $3x=8y-16$, to find the values of x and y in whole numbers.

Here, $x=\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}=$ *wh. or* $\frac{2y-1}{3}$ $=$ *wh.*

And, $\frac{2y-1}{3}\times2=\frac{4y-2}{3}=$ *wh. But* $\frac{3y}{3}$ *is alſo* $=$ *wh.*

Therefore $\frac{4y-2}{3}-\frac{3y}{3}=\frac{y-2}{3}=$ *wh.* $=p$.

Whence $y=3p+2$; *and by taking* $p=0$, *we ſhall have* $y=2$ *and* $x=0$, *the anſwer.*

3. Given $9x+13y=2000$; to find all the poſſible values of x and y in whole numbers.

Firſt, $x=\frac{2000-13y}{9}=222-y+\frac{2-4y}{9}=$ *wh.*

Or, by rejecting $222-y$, $\frac{2-4y}{9}=$ *wh. number.*

And, $\frac{2-4y}{9}\times2=\frac{4-8y}{9}=$ *wh. But,* $\frac{9y}{9}$ *is alſo* $=$ *wh.*

Therefore, $\frac{9y}{9}+\frac{4-8y}{9}=\frac{y+4}{9}=$ *wh.* $=p$.

Whence $y=9p-4$; *and, by taking* $p=1$, *we ſhall have* $y=5$ *and* $x=215$.

And, by adding 9, *continually to the laſt value of y, and ſubtracting* 13 *from that of x, all the poſſible anſwers will ſtand as follows:*

$$x=\left\{\begin{array}{l}215\cdot189\cdot163\cdot137\cdot111\cdot85\cdot59\cdot33\\202\cdot176\cdot150\cdot124\cdot\ 98\cdot72\cdot46\cdot20\end{array}\right.\cdot7$$

$$y=\left\{\begin{array}{l}5\cdot23\cdot41\cdot59\cdot77\cdot\ 95\cdot113\cdot131\\14\cdot32\cdot50\cdot68\cdot86\cdot104\cdot122\cdot140\end{array}\right.\cdot149$$

4. Given $24x=13y+16$, to find x and y in whole numbers. *Anſ.* $x=5$, *and* $y=8$.

5. Given $14x=5y+7$, to find x and y in whole numbers. *Anſ.* $x=3$, *and* $y=7$.

6. Given $27x=1600-16y$, to find x and y in whole numbers. *Anſ.* $y=19$, *and* $x=48$.

7. Given $87x+256y=15410$, to find the leaſt value of x, and the greateſt of y, in whole poſitive numbers. *Anſ.* $x=30$, *and* $y=$ [illegible]. 50

8. Given $5x+7y+11z=224$, to find all the poſſible values of x, y, and z, in whole numbers.
Anſ. The number of anſwers is 60.

9. How many different ways is it poſſible to pay 100l. in guineas and piſtoles only; a guinea being equal to 21 s. and a piſtole to 17 s.
Anſ. 6 *different ways.*

10. To determine whether it be poſſible to pay 100l. in guineas and moidores only.
Anſ. The queſtion is impoſſible.

11. Forty-one perſons, men, women and children, ſpent among them 40 s. of which each man paid 4 s. each woman 3 s. and each child 4 d. how many were there of each? *Anſ.* 5 *men,* 3 *women, and* 33 *children.*

12. I owe my friend a ſhilling, and have nothing about me but guineas, and he has nothing but louis d'ors; the queſtion is, how muſt I acquit myſelf of the debt? the louis d'ors being valued at 17 s. *Anſ. I muſt pay him* 13 *guineas and he muſt give me* 16 *louis d'ors.*

13. To find in what year of Chriſt the cycle of the ſun was 8, the cycle of the moon 10, and the cycle of indiction 10. *Anſ. In the year* 1567.

14. It is required to diſcharge a debt of 351 l. with guineas and moidores only, ſo that there may be the leaſt number of pieces of each ſort; and to find what the whole will amount to, when paid every way it will poſſibly admit of. *Anſ. The number of ways is* 36, *and the whole amount* 12636 *l.*

15. A vintner has wine at 2s. 1s. 10d. and 1s. 6d. *per* gallon: how much of each ſort muſt he take, ſo as to make a mixture of 30 gallons, to be ſold at 1s. 8d. *per* gallon? *Anſ.* 16, 2, 12; 17, 4, 9; 18, 6, 6; *or* 19, 8, 3, *of each ſort.*

16. To determine how many ways it is poſſible to pay 1000l. in crowns, guineas, and moidores only. *Anſ.* 70734 *different ways.*

PROBLEM II.

To find ſuch a whole number x, as being divided by the given numbers a, b, c, &c. ſhall leave the given remainders f, g, h, &c.

RULE.

1. Subtract each of the remainders from x, and divide the difference by a, and there will reſult $\frac{x-f}{a}$, $\frac{x-g}{a}$, $\frac{x-h}{a}$, &c. = whole numbers.

2. Call the value of x, in the firſt fraction, p, and ſubſtitute this quantity in the place of x in the ſecond fraction.

3. Find the leaſt value of p, in the ſecond fraction, by the laſt problem, and call it r.

4. Let the value of x be found in terms of r, and ſubſtitute this quantity in the place of x in the third fraction.

5. Find the leaſt value of r in the third fraction, and call it s; and the value of x in terms of s, being ſubſtituted for x in the fourth fraction, and ſo on, will give the whole number required.

EXAMPLES.

1. To find the leaſt whole number, which, being divided by 17, ſhall leave a remainder of 7; but, being divided by 26, the remainder ſhall be 13.

Let $x =$ *number required.*

Then $\frac{x-7}{17}$, *and* $\frac{x-13}{26} =$ *whole numbers.*

And, by putting $\frac{x-7}{17} = p$, *we ſhall have* $x = 17p + 7$;

Which value of x, *being ſubſtituted in the 2d fraction,*

gives $\frac{17p-6}{26} = wh$. *But* $\frac{26p}{26}$ *is alſo* $= wh$.

And, therefore, $\frac{26p}{26} - \frac{17p-6}{26} = \frac{9p+6}{26} = wh$.

or $\frac{9p+6}{26} \times 3 = \frac{27p+18}{26} = p + \frac{p+18}{26} = wh.$ *number.*

And, by rejecting p, *we have* $\frac{p+18}{26} = wh. = r$.

Hence $p = 26r - 18$, *and by taking* $r = 1$, *we ſhall have* $p = 8$.

And, conſequently $x = 17 \times 8 + 7 = 143$, *the number required.*

2. To find a number which being divided by 11, 19, and 29, the remainders ſhall be 3, 5, and 10.

Let $x =$ *number required.*

Then $\frac{x-3}{11}$, $\frac{x-5}{19}$ *and* $\frac{x-10}{29} =$ *whole numbers.*

And, by putting $\frac{x-3}{11} = p$, *we ſhall have* $x = 11p + 3$;

Which value of x, being ſubſtituted in the 2d fraction, gives $\frac{11p-2}{19}$ = *wh. or* $\frac{11p-2}{19}\times 2=\frac{22p-4}{19}=p+\frac{3p-4}{19}$ = *wh. And, by rejecting p, we ſhall have* $\frac{3p-4}{19}$ = *wh. Alſo* $\frac{3p-4}{19}\times 6=\frac{18p-24}{19}=\frac{18p-5}{19}-1$ = *wh. or by rejecting the* 1, $\frac{18p-5}{19}$ = *wh. number.*

But $\frac{19p}{19}$ *is, likewiſe,* = *wh. and, therefore,* $\frac{19p}{19}-\frac{18p-5}{19}=\frac{p+5}{19}$ = *wh. number, which let be put* = *r.*

Then, $p=19r-5$, *and* $x=(19r-5)\times 11+3=209r-52$.

And, by ſubſtituting this value of x in the 3*d fraction, we ſhall have* $\frac{209r-62}{29}=7r-2+\frac{6r-4}{29}$ = *wh. which by neglecting* $7r-2$ *gives* $\frac{6r-4}{29}$ = *wh. number.*

But, $\frac{6r-4}{29}\times 5=\frac{30r-20}{29}=r+\frac{r-20}{29}$ = *wh. or by rejecting r,* $\frac{r-20}{29}$ = *wh. which let be put* = *s.*

Then, $r=29s+20$; *and, by putting* $s=0$, *we ſhall have* $r=20$.

And, conſequently, $x=209\times 20-52=4128=$

Therefore, 4128 = *number required.*

3. To find the l[illegible] whole number, which, being divided by 19, ſhall leave a remainder of 7; but, being divided by 28, the remainder ſhall be 13.

Anſ. 349.

4. To find a number which, being divided by 3, 5, 7, and 2, will leave the remainders 2, 4, 6, and 0, reſpectively. *Anſ.* 104.

5. To find the leaſt whole number, which, being divided by 16, 17, 18, 19, and 20, ſhall leave 6, 7, 8, 9, and 10 remainders. *Anſ.* 232550.

6. To find the leaſt whole number, which, being divided by the nine digits reſpectively, ſhall leave no remainders. *Anſ.* 2520.

7. To find the leaſt whole number, which, being divided by 2, 3, 5, 7, and 11, ſhall leave 1, 2, 3, 4, and 5, for remainders. *Anſ.* 1523.

DIOPHANTINE PROBLEMS.

* *Diophantine problems* are thoſe which relate to the finding of ſquare and cube numbers, &c. and are ſuch as are generally capable of a great variety of anſwers. They are ſo called from their inventor *Diophantus* of *Alexandria* in *Egypt*, who flouriſhed in or about the third century, and is the firſt writer on Algebra we meet with amongſt the ancients.

Theſe queſtions are ſo exceedingly curious and abſtruſe, that nothing leſs than the moſt refined Algebra, applied with the utmoſt ſkill and judgment, can ſurmount the difficulties which attend them. And, in this way, no man has ever extended the limits of the analytic art further than *Diophantus*, or diſcovered greater penetration and judgment in the application of it.

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* That *Diophantus* was not the inventor of algebra, as has been generally imagined, is extremely obvious; for this method of applying it is ſuch, as could only have been uſed in a very advanced ſtate of the ſcience.

He no where ſpeaks of the fundamental rules and principles, as an inventor certainly would have done, but treats of it as an art already ſufficiently known; and ſeems to intend, not ſo much to teach it, as to cultivate and improve it, by ſolving ſuch queſtions as, before his time, had been thought too difficult to be ſurmounted.

When we conſider his work with attention, we are at a loſs which to admire moſt, his wonderful ſagacity and peculiar artifice, in forming ſuch poſitions as the nature of the problems required, or the more than ordinary ſubtilty of his reaſoning upon them.

Every particular queſtion puts us upon a new way of thinking, and furniſhes a freſh vein of analytical treaſure, which cannot but be very inſtructive to the mind, in conducting it through almoſt all difficulties of this kind, wherever they occur.

The following method of reſolving theſe queſtions will be found of conſiderable ſervice; but no general rule can be given, that will ſuit all caſes; and therefore the ſolution muſt often be left to the ſagacity and ſkill of the learner.

It is probable, therefore, that Algebra was known in the world long before the time of *Diophantus*; but that the works of preceding writers have been deſtroyed by the ravages of time, or the depredations of ignorant barbarians.

His *Arithmetics*, out of which theſe problems were moſtly collected, conſiſted originally of thirteen books; but the firſt ſix only are now extant. The beſt edition is ſaid to be that publiſhed at *Paris*, by Monſieur *Bachet*, in the year 1621. And in this work, the ſubject is ſo ſkilfully handled, that the moderns, notwithſtanding their other improvements, have been able to do little more than explain and illuſtrate his method.

Thoſe who have ſucceeded beſt, in this reſpect, are *Vieta*, *Brauncker*, *Kerſey*, *De Billy*, *Ozanam*, *Preſtet*, *Saunderſon*, *Fermat*, and *Euler*. The laſt of whom, in particular, has amplified and illuſtrated the Diophantine Algebra in as clear and ſatisfactory a manner as the ſubject ſeems to admit.

RULE.

1. For the root of the ſquare or cube required, put one or more letters ſuch, that when they are involved, either the given number, or the higheſt power of the unknown quantity, may vaniſh from the equation; and then, if the unknown quantity be but of one dimenſion, the problem will be ſolved by reducing the equation.

2. But if the unknown quantity be ſtill a ſquare, or a higher power, ſome other new letters muſt be aſſumed to denote the root; with which proceed as before; and ſo on till the unknown quantity be but of one dimenſion; and from this all the reſt will be determined.

EXAMPLES.

1. [*] To divide a given ſquare number (100) into two ſuch parts, that each of them may be a ſquare number.

Let x^2 ($=\square$) *be one of the parts, and then* $100-x^2$ *will be the other part, which is alſo to be a ſquare number.*

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[*] If $x-10$ had been made the ſide of the ſecond ſquare, in this queſtion, inſtead of $2x-10$, the equation would have been $x^2-20x+100=150-x^2$; in which caſe, x, the ſide of the firſt ſquare, would have been found $=10$, and $x-10$, or the ſide of the ſecond ſquare $=0$; and for this reaſon the ſubſtitution, $x-10$, was avoided; but $3x-10$, $4x-10$, or any other quantity of the ſame kind, would have ſucceeded equally as well as the former; though, in ſome caſes, the reſults would have been leſs ſimple.

Aſſume the ſide of this ſecond ſquare $=2x-10$.
Then will $100-x^2=(2x-10)^2=4x^2-40x+100$;
And, by reduction, $x=8$, *and* $2x-10=6$.
Therefore 64 *and* 36 *are the parts required.*

THE SAME GENERALLY.

Let $a^2=$ *given ſquare number,* $x^2(=\square)=$ *one of its parts, and* $a^2-x^2=$ *the other, which is alſo to be a ſquare number.*

Aſſume the ſide of this ſecond ſquare $=rx-a$,
then will $a^2-x^2=(rx-a)^2=r^2x^2-2arx+a^2$;

And, by reduction, $x=\dfrac{2ar}{r^2+1}$, *and* $rx-a=\dfrac{2ar^2}{r^2+1}-a$ $=\dfrac{2ar^2}{r^2+1}-\dfrac{ar^2+a}{r^2+1}=\dfrac{ar^2-a}{r^2+1}=$ *to a ſquare number.*

Therefore $\left(\dfrac{2ar}{r^2+1}\right)^2$ *and* $\left(\dfrac{ar^2-a}{r^2+1}\right)^2$ *are the parts required; where a and r may be any numbers, taken at pleaſure.*

2. * To divide a given number (13) conſiſting of two known ſquare numbers (9 and 4) into two other ſquare numbers.

If s and r be any two unequal numbers, of which s is the greater; then will $2rs$, s^2-r^2 and s^2+r^2, be the perpendicular, baſe, and hypothenuſe of a right-angled triangle.

And from this canon two ſquare numbers may be found, whoſe ſum or difference ſhall be ſquare numbers; for $(2rs)^2+(s^2-r^2)^2=(s^2+r^2)^2$, and $(r^2+s^2)^2-(2rs)^2=(s^2-r^2)^2$, or $(s^2+r^2)^2-(s^2-r^2)^2=(2rs)^2$; and this when s and r are any numbers whatever.

* This queſtion is conſidered by *Diophantus* as a very important one, being made the foundation, or *baſis*, of moſt

For the ſide of the firſt ſquare ſought, put $rx-3$; *and for the ſide of the ſecond,* $sx-2$; *r being the greater number and s the leſs.*

Then will $(rx-3)^2+(sx-2)^2=(r^2x^2-6rx+9)+(s^2x^2-4sx+4)=(r^2+s^2x^2)-(6r-4sx)+13=13$, *or* $(r^2+s^2)x^2=(6r+4s)x$.

And from this equation, by reduction, x is found $=\frac{6r+4s}{r^2+s^2}$

Whence, $rx-3=\frac{6r^2+4rs}{r^2+s^2}-3=\frac{3r^2+4rs-3s^2}{r^2+s^2}=$ *ſide of the firſt ſquare ſought.*

And, $sx-2=\frac{6rs+4s^2}{r^2+s^2}-2=\frac{6rs-2r^2+2s^2}{r^2+s^2}=$ *ſide of the ſecond.*

So that if r be taken $=2$, *and* $s=1$, *we ſhall have* $\frac{3r^2+4rs-3s^2}{r^2+s^2}=\frac{17}{5}$, *and* $\frac{6rs-2r^2+2s^2}{r^2+s^2}=\frac{6}{5}$ *for the ſides of the ſquares, in numbers, as was required.*

If a^2+b^2 *be put equal to the number to be divided, the general ſolution may be given in exactly the ſame manner.*

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of his other problems.——In the ſolution of it, given above, the values of *r* and *s* may be taken equal to any numbers whatever, provided the proportion of thoſe numbers be not the ſame as that of 3 (*a*) to 2 (*b*), or 3+2 (*a*+*b*) to 3—2 (*a*—*b*). And the reaſon of this reſtriction is, that, if *r* and *s* were ſo taken, the ſides of the ſquares ſought would come out the ſame as the ſides of the known ſquares which compoſe the given number, and therefore the operation would be uſeleſs.

The excellent old *Kerſey*, after amplifying and illuſtrating this problem in a variety of ways, concludes his chapter thus: "For a further account of this *rare* ſpeculation, ſee *Anderſonus*, Theorem 2, of *Vieta's myſterious* doctrine of Angular Sections; and likewiſe *Herigonius*, at the latter end of the firſt *tome* of his *Curſus Mathematicus*."

3. To find two ſquare numbers, whoſe difference ſhall be equal to any given number (d).

Let d be reſolved into any two unequal factors a and b; a being the greater and b the leſs.

Alſo put x for the ſide of the leſſer ſquare ſought, and $x+b=$ ſide of the greater.

Then $(x+b)^2-x^2=x^2+2bx+b^2-x^2=2bx+b^2=d$ (ab).

And if this be divided by b, we ſhall have $2x+b=a$.

Whence, $x=\frac{a-b}{2}=$ *ſide of the leſſer ſquare ſought,*

and $x+b=\frac{a-b}{2}+b=\frac{a+b}{2}=$ *ſide of the greater.*

So that by putting $d=60$, and $a\times b=2\times 30$, we ſhall have $\frac{30-2}{2}=14$, *and* $\frac{30+2}{2}=16$, *or* $(14)^2=196$, *and* $(16)^2=256$ *for the ſquares in numbers; and ſo far any difference or factors whatever.*

4. To find two numbers ſuch, that, if either of them be added to the ſquare of the other, the ſum ſhall be a ſquare number.

Let the numbers ſought be x and y.

then $x^2+y=\square$, *and* $y^2+x=\square$.

And, if $r-x$ be aſſumed for the ſide of the firſt ſquare x^2+y, we ſhall have $x^2+y=r^2-2rx+x^2$, *or* $y=r^2-2rx$.

Whence $2rx=r^2-y$, *or* $x=\frac{r^2-y}{2r}$.

Again, if $y+s$ be aſſumed for the ſide of the ſecond ſquare, we ſhall have $y^2+\frac{r^2-y}{2r}(y^2+x)=(y+s)^2=y^2+2sy+s^2$.

Whence, $\frac{r^2-y}{2r}=2sy+s^2$, *or* $r^2-y=4rsy+2rs^2$.

O

And conſequently, $y=\frac{r^2-2rs^2}{4rs+1}$ *and* $x=\frac{r^2-y}{2r}=\frac{2r^2s+s^2}{4rs+1}$.

So that $\frac{r^2-2rs^2}{4rs+1}$ *and* $\frac{2r^2s+s^2}{4rs+1}$ *are the numbers required; where* r *and* s *may be taken at pleaſure, provided* r *be greater than* $2s^2$.

5. To find two numbers, whoſe ſum and difference ſhall be both ſquare numbers.

Let x *and* x^2-x *be the two numbers ſought.*

Then, ſince their ſum is evidently a ſquare number, one of the conditions of the queſtions will be anſwered.

There remains, therefore, only their difference x^2-2x *to be made a ſquare.*

And, if for the ſide of this ſquare there be put $x-r$*, we ſhall have* $x^2-2rx+r^2=x^2-2x$*, or* $2rx-2x=r^2$.

Whence, $x=\frac{r^2}{2r-2}$ *and* $x^2-x=\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$.

So that $\frac{r^2}{2r-2}$ *and* $\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$ *are the numbers required; where* r *may be taken at pleaſure, provided it be greater than* 1.

6. To find three numbers ſuch, that not only the ſum of all three of them, but alſo the ſum of every two, ſhall be a ſquare number.

Let $4x$, x^2-4x *and* $2x+1$ *be the three numbers ſought.*

Then $(4x)+(x^2-4x)=x^2$, $(x^2-4x)+(2x+1)=x^2-2x+1$, *and* $(4x+x^2-4x+2x+1)=x^2+2x+1$, *are evidently ſquares.*

And, therefore, three of the conditions mentioned in the queſtions are accompliſhed.

Whence it remains only to make the quantity $(4x)+(2x+1)$, *or* $6x+1=$ *to a ſquare.*

Let, therefore, $6x+1=a^2$; *and we ſhall have* $x=\frac{a^2-1}{6}$.

And, conſequently, $\frac{4a^2-4}{6}$, $\left(\frac{a^2-1}{6}\right)^2-\frac{4a^2-4}{6}$, *and* $\frac{2a^2-2}{6}+1$; *or* $\frac{2a^2-2}{3}$, $\frac{a^4-26a^2+25}{36}$, *and* $\frac{a^2+2}{3}$ *are the numbers required; where a may be taken at pleaſure, provided it be greater than* 5.

7. To find three ſquare numbers ſuch, that the ſum of every two of them ſhall be a ſquare number *.

Let x^2, y^2, *and* z^2 *be the numbers ſought;*

Then $x^2+z^2=\square$, $y^2+z^2=\square$, *and* $x^2+y^2=\square$.

Or $\frac{x^2}{z^2}+1=\square$, $\frac{y^2}{z^2}+1=\square$, *and* $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\square$.

And, by putting $\frac{x}{z}=\frac{s^2-1}{2s}$, *and* $\frac{y}{z}=\frac{r^2-1}{2r}$,

we ſhall have $\frac{x^2}{z^2}+1=\frac{s^4+2s^2+1}{4s^2}$, *and* $\frac{y^2}{z^2}+1=\frac{r^4+2r^2+1}{4r^2}$ *which are both evidently ſquares; and therefore it remains only to make* $\frac{x^2}{z^2}+\frac{y^2}{z^2}=$ *ſquare number.*

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* This queſtion is capable of a great variety of anſwers; but the leaſt roots, which have yet been found, in whole numbers, are 44, 117, and 240. See *Elémens d'Algebre*, par *M. Euler*, tome II. page 327; which is a work particularly calculated for the uſe of thoſe who wiſh to obtain a knowledge of Algebra without the aſſiſtance of a maſter.

But $\frac{x^2}{z^2}+\frac{y^2}{z^2}=(\frac{s^2-1}{2s})^2+(\frac{r^2-1}{2r})^2=\frac{(s^2-1)^2}{4s^2}+\frac{(r^2-1)^2}{4s^2}=\frac{4r^2\times(s^2-1)^2+4s^2\times(r^2-1)^2}{4r^2s^2}=$ *to a ſquare number.*

Or $r^2\times(s^2-1)^2+s^2\times(r^2-1)^2=r^2\times(s+1)^2\times(s-1)^2+s^2\times(r+1)^2\times(r-1)^2=$ *to a ſquare number.*

And, by making $r-1=s+1$, *or* $r=s+2$, *we ſhall have* $(s+2)^2\times(s+1)^2\times(s-1)^2+s^2\times(s+1)^2\times(s+1)^2=$ *to a ſquare number.*

Or $(s+2)^2\times(s-1)^2+s^2\times(s+3)^2=2s^4+8s^3+6s^2-4s+4=$ *to a ſquare number.*

Now, let the root of this ſquare be aſſumed $=\frac{5}{4}s^2-s+2$,

Then, $2s^4+8s^3+6s^2-4s+4=(\frac{5}{4}s^2-s+2)^2=\frac{25}{16}s^4-\frac{5}{2}s^3+5s^2+s^2-4s+4$; *or* $2s^4+8s^3=\frac{25}{16}s^4-\frac{5}{2}s^3$; *or* $2s+8=\frac{25}{16}s-\frac{5}{2}$.

Whence $s=-24$, *and* $r=-22$.

And, conſequently, $\frac{x}{z}=\frac{s^2-1}{2s}=-\frac{575}{48}$, *and* $\frac{y}{z}=\frac{r^2-1}{2r}=-\frac{483}{44}$;

or $x=-\frac{575z}{48}$, *and* $y=-\frac{483z}{44}$.

In order, therefore, to have the anſwer in whole numbers, let $z=528$, *and we ſhall have* $x=6325$, *and* $y=5796$, *or* 528, 5796 *and* 6325 *for the roots of the ſquares required.*

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8. To find a number x ſuch, that $x+1$ and $x-1$ ſhall be both ſquare numbers. *Anſ.* $x=\frac{5}{4}$.

9. To find a number x ſuch, that $x+128$ and $x+192$ ſhall be both ſquares. *Anſ.* $x=97$.

10. To find a number x ſuch, that x^2+x and x^2-x may be both ſquares. *Anſ.* $x=\frac{25}{24}$.

11. To find two numbers x and y ſuch, that $x+y$, x^2+y and y^2+x may be all ſquares.

Anſ. $x=\frac{1}{6}$ *and* $y=\frac{1}{12}$.

12. To find three ſquare numbers in arithmetical progreſſion. *Anſ.* 1, 25, *and* 49.

13. To find three ſquare numbers in harmonical proportion. *Anſ.* 1225, 49, *and* 25.

14. To find three numbers in arithmetical progreſſion ſuch, that the ſum of every two of them may be a ſquare number. *Anſ.* $120\frac{1}{2}$, $840\frac{1}{2}$, *and* $1560\frac{1}{2}$.

15. To find three numbers ſuch, that, if to the ſquare of every one of them the ſum of the other two be added, the three ſums ſhall be all ſquares.

Anſ. $\frac{8}{3}$, $\frac{16}{3}$ *and* 1.

16. To find two numbers in proportion as 8 is to 5, and ſuch that the ſum of their ſquares ſhall make a ſquare number. *Anſ.* 576 *and* 1080.

17. To find four numbers ſuch, that, if a ſquare number (100) be added to the product of every two of them, the ſums ſhall be all ſquares.

Anſ. 2, 32, 88, *and* 168.

18. To find two numbers ſuch, that their difference may be equal to the difference of their ſquares, and that the ſum of their ſquares ſhall be a ſquare number.

Anſ. $\frac{4}{7}$ *and* $\frac{3}{7}$.

19. To find three numbers in geometrical proportion ſuch, that every one of them being increaſed by a given number (19) ſhall make ſquare numbers.

Anſ. 81, $\frac{5}{4}$ *and* $\frac{25}{1296}$.

20. To find two numbers ſuch, that, if their product be added to the ſum of their ſquares, it ſhall make a ſquare number.

Anſ. 5 *and* 3, 8 *and* 7, 16 *and* 5, *&c.*

21. To divide a given number (10) into four ſuch parts, that the ſum of every three of them may make a ſquare number. *Anſ.* 1, 6, $\frac{186}{289}$ *and* $\frac{681}{289}$.

22. * To find three ſquare numbers ſuch, that their ſum being ſeverally added to their three ſides ſhall make ſquare numbers.

Anſ. $\frac{4418}{62920}$, $\frac{11254}{62920}$, *and* $\frac{19881}{62920}$ = *roots required.*

23. To find two numbers ſuch, that their ſum being increaſed and leſſened, either by their difference. or the difference of their ſquares, the ſums and remainders ſhall be all ſquares. *Anſ.* $\frac{49}{50}$ *and* $\frac{1}{50}$.

24. To find two numbers ſuch, that not only each number, but alſo their ſum and their difference, being increaſed by unity, ſhall be all ſquare numbers.

Anſ. 3024 *and* 5624.

25. To find three numbers ſuch, that whether their ſum be added to, or ſubtracted from, the ſquare of each particular number, the numbers thence ariſing ſhall be all ſquares. *Anſ.* $\frac{406}{96}$, $\frac{518}{96}$ *and* $\frac{791}{96}$.

26. To find three ſquare numbers ſuch, that the ſum of their ſquares ſhall alſo be a ſquare number.

Anſ. 9, 16, *and* $\frac{144}{25}$.

27. To find three ſquare numbers ſuch, that the difference of every two of them ſhall be a ſquare number. *Anſ.* 485809, 34225 *and* 23409.

28. To divide any given cube number (8) into three other cube numbers. *Anſ.* $\frac{64}{27}$, $\frac{125}{27}$ *and* 1.

29. Two cube numbers (8 and 1) being given, to find two other cube numbers, whoſe difference ſhall be equal to the ſum of the given cubes.

Anſ. $\frac{8000}{343}$, *and* $\frac{4913}{343}$.

30. To divide a given number (28) compoſed of two cube numbers (27 and 1) into two other cube numbers. *Anſ.* $\frac{63284705}{21446828}$ *and* $\frac{28340511}{21446828}$ *the roots.*

31. To find three cube numbers ſuch, that, if from every one of them a given number (1) be ſubtracted, the ſum of the remainders ſhall be a ſquare.

Anſ. $\frac{4913}{3375}$, $\frac{21952}{3375}$ *and* 8.

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* The anſwers to many of theſe queſtions cannot be given in whole numbers.

32. To find three numbers ſuch, that, if they be ſeverally added to the cube of their ſum, the three ſums thence ariſing ſhall be all cubes.

Anſ. $\frac{15338}{157464}$, $\frac{18577}{157461}$ *and* $\frac{23675}{157464}$.

33. To find three numbers in arithmetical proportion ſuch, that the ſum of their cubes ſhall be a cube.

Anſ. 3, 4, 5, *or* 149, 256, 363, *&c.*

34. To find three cube numbers ſuch, that their ſum ſhall be a cube number.

Anſ. 3^3, 4^3, *and* 5^3, *or* 21^3, 19^3, 18^3, *&c.*

35. To find two numbers ſuch, that their ſum ſhall be equal to the ſum of their cubes. *Anſ.* $\frac{5}{7}$ *and* $\frac{8}{7}$.

OF THE
SUMMATION AND INTERPOLATION
OF
INFINITE SERIES.

The *doctrine of infinite ſeries* is a ſubject which has engaged the attention of the greateſt mathematicians in all ages; and is, perhaps, one of the moſt abſtruſe and difficult branches of abſtract mathematics.

To find the ſum of a ſeries, the number of whoſe terms is inexhauſtible, or infinite, has been conſidered by ſome as an extravagant paradox, or a thing impoſſible to be done. But this difficulty will be eaſily removed, by conſidering that every finite magnitude whatever is diviſible *in infinitum*, or conſiſts of an infinite number of parts, whoſe aggregate, or ſum, is equal to the quantity firſt propoſed.

A number actually infinite is, indeed, a plain contradiction to all our ideas; for any number which we can poſſibly conceive, or of which we have any notion, muſt always be determinate and finite; ſo that a greater may be ſtill aſſigned, and a greater after this;

and ſo on, without a poſſibility of ever coming to an end of the increaſe or addition.

And this inexhauſtibility, in the nature of numbers, is, therefore, all that we can diſtinctly comprehend by their infinity; for though we can eaſily conceive that a finite quantity may become greater and greater without end, yet we are not from thence enabled to form any notion of the *ultimatum*, or laſt magnitude, which is incapable of further augmentation.

We cannot, therefore, apply to an infinite ſeries the common notion of a ſum, or a collection of ſeveral particular numbers, which are joined and added together, one after another; for this ſuppoſes that thoſe particulars are all known and determined. But as every ſeries generally obſerves ſome regular law, and continually approaches towards a term or limit, we can eaſily conceive it to be a whole, of its own kind, and that it muſt have a certain real value, whether that value be determinable or not.

Thus, in many ſeries, a number is aſſignable, beyond which no number of its terms can ever reach, or indeed ever be equal to it; but yet may approach to it in ſuch a manner as to want leſs than any given difference. And this we may call the value or ſum of the ſeries; not as being a number found by the common method of addition, but as ſuch a limitation of the value of the ſeries, taken in all its infinite capacity, that if it were poſſible to add all the terms together, one after another, the ſum would be equal to that number.

Again, in other ſeries, the value has no limitation; and this may be expreſſed by ſaying, that the ſum of the ſeries is infinitely great; or, which is the ſame thing, that it has no determinate and aſſignable value; but may be carried on to ſuch a length, that its ſum ſhall exceed any given number whatever.

According to the common rule for ſumming up a finite progreſſion, of a geometric decreaſing ſeries, where r is the ratio, l the firſt term, and a the leaſt, the ſum is $(rl-a)\div(r-1)$: and if we ſuppoſe a, the leſs extreme, to be actually decreaſed to 0, then the ſum of the whole ſeries will be $rl\div(r-1)$: for it is demonſtrable, that the ſum of no aſſignable number of terms of the ſeries can ever be equal to that quotient; and yet no number leſs than it will ever be equal to the value of the ſeries.

Whatever conſequences, therefore, follow from the ſuppoſition of $rl\div(r-1)$ being the true and adequate value of the ſeries, taken in all its infinite capacity, as if all the parts were actually determined, and added together, they can never be the occaſion of any aſſignable error, in any operation or demonſtration where it is uſed in that ſenſe; becauſe, if you ſay that it exceeds that value, it is demonſtrable that this exceſs muſt be leſs than any aſſignable difference, which is, in effect, no difference at all; and, therefore, the ſuppoſed error cannot exiſt, and conſequently $rl\div(r-1)$ may be looked upon as expreſſing the adequate and juſt value of the ſeries, continued to infinity.

But we are further ſatisfied of the reaſonableneſs of this doctrine, by finding, in fact, that a finite quantity actually converts into an infinite ſeries, as appears in the caſe of circulating decimals. Thus, $\frac{2}{3}$ turned into a decimal is $=.6666$, &c. $=\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}$, &c, continued *ad infinitum*. But this is plainly a geometric ſeries, beginning from $\frac{6}{10}$, in the continued ratio of 10 to 1; and the ſum of all its terms, continued to infinity, will evidently be equal to $\frac{2}{3}$, or the number from whence it was originally derived.

And the ſame may be ſhewn of many other ſeries, and of all circulating decimals in general.

PROBLEM I.

Any ſeries being given to find the ſeveral orders of differences.

RULE.

1. Take the firſt term from the ſecond, the ſecond from the third, the third from the fourth, &c. and the remainders will form a new ſeries, called the *firſt order of differences.*

2. Take the firſt term of this laſt ſeries from the ſecond, the ſecond from the third, the third from the fourth, &c. and the remainders will form another new ſeries, called the *ſecond order of differences.*

3. Proceed, in like manner, for the *third, fourth, fifth,* &c. orders of differences; and ſo on till they terminate, or are carried as far as is thought neceſſary.

EXAMPLES.

1. To find the ſeveral orders of differences in the ſeries 1, 4, 9, 16, 25, 36, &c.

1, 4, 9, 16, 25, 36, &c.
3, 5, 7, 9, 11, &c. *1ſt diff.*
2, 2, 2, 2, &c. *2d diff.*
0, 0, 0, &c.

2. To find the ſeveral orders of differences in the ſeries 1, 8, 27, 64, 125, 216, &c.

1, 8, 27, 64, 125, 216, &c.
7, 19, 37, 61, 91, &c. *1ſt diff.*
12, 18, 24, 30, &c. *2d diff.*
6, 6, 6, &c. *3d diff.*
0, 0, &c.

3. To find the ſeveral orders of differences in the ſeries 1, 3, 6, 10, 15, 21, &c.

Anſ. *1ſt* 2, 3, 4, 5, *&c.* *2d* 1, 1, 1, *&c.*

4. To find the ſeveral orders of differences in the ſeries 1, 6, 20, 50, 105, 196, &c.

Anſ. *1ſt* 5, 14, 30, 55, 91, *&c.* *2d* 9, 16, 25, 36, *&c.* *3d* 7, 9, 11, *&c.* *4th* 2, 2, *&c.*

PROBLEM II.

Any ſeries, a, b, c, d, e, &c. being given, to find the firſt term of the nth order of differences.

RULE *.

Let δ ſtand for the firſt term of the *nth.* differences.

Then will $a - nb + n \times \frac{n-1}{2} c - n \times \frac{n-1}{2} \times \frac{n-2}{3} d + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} e$, &c. to $n+1$ terms $= \delta$, when n is an even number.

And $-a + nb - n \times \frac{n-1}{2} c + n \times \frac{n-1}{2} \times \frac{n-2}{3} d - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} e$, &c. to $n+1$ terms $= \delta$, when n is an odd number.

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* When the ſeveral orders happen to be very great, it will be more convenient to take the logarithms of the quantities concerned, whoſe differences will be ſmaller; and, when the operation is finiſhed, the quantity anſwering to the laſt logarithm may be eaſily found.

EXAMPLES *.

1. Required the firſt term of the third order of differences of the ſeries 1, 5, 15, 35, 70, &c.

Let $a, b, c, d, e,$ *&c.* $= 1, 5, 15, 35, 70,$ *&c. and* $n = 3$.

Then $-a + nb - n \times \frac{n-1}{2} c + n \times \frac{n-1}{2} \times \frac{n-2}{3} d = -a + 3b - 3c + d = -1 + 15 - 45 + 35 = 4 =$ *the firſt term required.*

2. Required the firſt term of the fourth order of differences of the ſeries, 1, 8, 27, 64, 125, &c.

Let $a, b, c, d, e,$ *&c.* $= 1, 8, 27, 64, 125,$ *&c. and* $n = 4$.

Then $a - nb + n \times \frac{n-1}{2} c - n \times \frac{n-1}{2} \times \frac{n-2}{3} d + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} e, = a - 4b + 6c - 4d + e = 1 - 32 + 162 - 256 + 125 = 0$; *ſo that the firſt term of the fourth order is* 0.

3. Required the firſt term of the fifth order of differences of the ſeries 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c.

Anſ. $-\frac{1}{32}$

4. Required the firſt term of the 8th order of differences of the ſeries 1, 3, 9, 27, 81, &c. *Anſ.* 256.

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* The labour in theſe kind of queſtions may be often abridged, by putting cyphers for ſome of the terms at the beginning of the ſeries; by which means ſeveral of the differences will be equal to 0, and the anſwer, on that account, obtained in fewer terms.

PROBLEM III.

To find the nth. term of the ſeries, a, b, c, a, e, &c.

RULE.

Let d^{I}, d^{II}, d^{III}, d^{IV}, &c. be the firſt of the ſeveral orders of differences, found as in the laſt problem:

Then will $a+\frac{n-1}{1}d^{\mathrm{I}}+\frac{n-1}{1}\times\frac{n-2}{2}d^{\mathrm{II}}+\frac{n-1}{1}\times\frac{n-2}{2}\times\frac{n-3}{3}d^{\mathrm{III}}+\frac{n-1}{1}\times\frac{n-2}{2}\times\frac{n-3}{3}\times\frac{n-4}{4}d^{\mathrm{IV}}$, &c. be $=$ nth. term required.

EXAMPLES.

1. To find the 12th term of the ſeries, 2, 6, 12, 20, 30, &c.

2, 6, 12, 20, 30, &c.
4, 6, 8, 10, &c.
2, 2, 2, &c.
0, 0, &c.

Here 4 and 2 are the firſt terms of the differences;
Let, therefore, $4=d^{\mathrm{I}}$, $2=d^{\mathrm{II}}$, *and* $n=12$.

Then $a+\frac{n-1}{1}d^{\mathrm{I}}+\frac{n-1}{1}\times\frac{n-2}{2}d^{\mathrm{II}}=2+11d^{\mathrm{I}}+55d^{\mathrm{II}}=2+44+110=156=12th$ *term, or the anſwer required.*

2. Required the 20th term of the ſeries 1, 3, 6, 10, 15, 21, &c.

1, 3, 6, 10, 15, 21, &c.
2, 3, 4, 5, 6, &c.
1, 1, 1, 1, &c.
0, 0, 0, &c.

Here 2 *and* 1 *are the firſt terms of the differences.*
Let, therefore, $2=d^{\mathrm{I}}$, $1=d^{\mathrm{II}}$, *and* $n=20$,

Then $a+\frac{n-1}{1}d^{\mathrm{I}}+\frac{n-1}{1}\times\frac{n-2}{2}d^{\mathrm{II}}=1+19d^{\mathrm{I}}+171d^{\mathrm{II}}=1+38+171=210=$ 20*th term required.*

3. Required the 15th term of the ſeries 1, 4, 9, 16, 25, 36, &c. *Anſ.* 225.

4. Required the 20th term of the ſeries, 1, 8, 27, 64, 125, &c. *Anſ.* 8000.

PROBLEM IV.

To find the ſum of n terms of the ſeries a, b, c, d, e, &c.

RULE.

Let d^{I}, d^{II}, d^{III}, d^{IV}, &c. be the firſt of the ſeveral orders of differences.

Then will $na+n\times\frac{n-1}{2}d^{\mathrm{I}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{\mathrm{II}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d^{\mathrm{III}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}d^{\mathrm{IV}}$, &c. = to the ſum of n terms of the ſeries.

EXAMPLES.

1. * To find the ſum of n terms of the ſeries 1, 2, 3, 4, 5, 6, &c.

1, 2, 3, 4, 5, 6, &c.
1, 1, 1, 1, 1, &c.
0, 0, 0, 0, &c.

Here 1 *and* 0 *are the firſt terms of the differences.*
Let, therefore, $a=1$, $d^{\mathrm{I}}=1$, *and* $d^{\mathrm{II}}=0$;

Then will $na+n\times\frac{n-1}{2}d^{\mathrm{I}}=n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=$ *ſum of n terms, as required.*

2. To find the ſum of n terms of the ſeries 1^2, 2^2, 3^2, 4^2, 5^2, &c. or 1, 4, 9, 16, 25, &c.

1, 4, 9, 16, 25, &c.
3, 5, 7, 9, &c.
2, 2, 2, &c.
0, 0, &c.

Here 3 *and* 2 *are the firſt terms of the differences:*
Let, therefore, $a=1$, $d^{\mathrm{I}}=3$ *and* $d^{\mathrm{II}}=2$.

Then will $na+n\times\frac{n-1}{2}d^{\mathrm{I}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{\mathrm{II}}=n+3n\times\frac{n-1}{2}+n\times\frac{n-1}{2}\times\frac{n-2}{3}=\frac{3n^2-n}{2}+\frac{n^3-3n^2+2n}{3}$ $=\frac{n\times(n+1)\times(2n+1)}{6}=$ *ſum of n terms as required.*

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* Any term of a given ſeries, or the ſum of any number of its terms, may be accurately determined, when the differences of any order become at laſt equal to each other.

* 3. To find the ſum of n terms of the ſeries 1^3, 2^3, 3^3, 4^3, 5^3, &c. or 1, 8, 27, 64, 125, &c.

1, 8, 27, 64, 125, &c.
7, 19, 37, 61, &c.
12, 18, 24, &c.
6, 6, &c.
0, &c.

Here the firſt term of the differences are 7, 12 *and* 6. *Let, thereſore,* $a=1$, $d^{\text{I}}=7$, $d^{\text{II}}=12$, *and* $d^{\text{III}}=6$.

Then will $$na+n\times\frac{n-1}{2}d^{\text{I}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{\text{II}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d^{\text{III}}=n+7n\times\frac{n-1}{2}+12n\times\frac{n-1}{2}\times\frac{n-2}{3}+6n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}=\frac{7n^2-5n}{2}+2n^3-6n^2+4n+\frac{n^4-6n^3+11n^2-6n}{4}=\frac{n^4+2n^3+n^2}{4}$$ = *ſum of n terms, as required.*

4. To find the ſum of n terms of the ſeries 2, 6, 12, 20, 30, &c.

Anſ. $\frac{n\times(n+1)\times(n+2)}{3}$.

5. To find the ſum of n terms of the ſeries 1, 3, 6, 10, 15, &c.

Anſ. $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$.

6. To find the ſum of n terms of the ſeries 1, 4, 10, 20, 35, &c.

Anſ. $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$.

7. To find the ſum of n terms of the ſeries 1^4, 2^4, 3^4, 4^4, &c. or 1, 16, 81, 256, &c.

Anſ. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$.

PROBLEM V.

The ſeries a, b, c, d, e, &c. being given, whoſe terms are at an unit's diſtance from each other; to find any intermediate term by interpolation.

RULE.

Let x be the diſtance of any term y to be interpolated, and d^{I}, d^{II}, d^{III}, &c. the firſt terms of the differences:

Then will $a+xd^{I}+x\times\frac{x-1}{2}d^{II}+x\times\frac{x-1}{2}\times\frac{x-2}{3}d^{III}+x\times\frac{x-1}{2}\times\frac{x-2}{3}\times\frac{x-3}{4}d^{IV}$ &c. $=y$.

EXAMPLES.

1. Given the logarithmic ſines of 1° 0′, 1° 1′, 1° 2′, and 1° 3′, to find the ſine of 1° 1′ 40″.

	1° 0′	1° 1′	1° 2′	1° 3′
Sines	8.2418553	8.2490332	8.2560943	8.2630424
		71779	70611	69481
			−1168	−1130
				38

Here the firſt terms of the differences are 71779, −1168, *and* 38.

Let, therefore, $x=1°\ 1'\ 40''-1°\ 0'=1'\ 40''=1\frac{2}{3}=$ *diſtance of y, the term to be interpolated; and* $d^{I}=71779$, $d^{II}=-1168$, *and* $d^{III}=38$.

Then will $y=a+xd^{I}+x\times\frac{x-1}{2}d^{II}+x\times\frac{x-1}{2}\times\frac{x-2}{3}d^{III}=a+\frac{5}{3}d^{I}+\frac{5}{9}d^{II}-\frac{5}{81}d^{III}=8.2418553+$

$.0119631 - .0000694 - .0000002 = 8.2537533 =$ *ſine of* $1^\circ\ 1'\ 40''$, *as was required.*

2. Given the ſeries $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$, &c. to find the term which ſtands in the middle between $\frac{1}{52}$ and $\frac{1}{53}$.

Anſ. $\frac{2}{105}$.

3. Given the natural tangents of 88°, 54', 88°, 55', 88°, 56', 88°, 57', 88°, 58, and 88°, 59; to find the tangent of 88°, 58', 18". *Anſ.* 55.711144.

PROBLEM VI.

Having given a ſeries of equidiſtant terms, a, b, c, d, e, &c. whoſe firſt differences are ſmall; to find any intermediate term by interpolation.

RULE *.

Find the value of the unknown quantity in the equation which ſtands againſt the given number of terms, in the following table, and it will give the term required.

1. $a - b = 0$
2. $a - 2b + c = 0$
3. $a - 3b + 3c - d = 0$
4. $a - 4b + 6c - 4d + e = 0$
5. $a - 5b + 10c - 10d + 5e - f = 0$
6. $a - 6b + 15c - 20d + 15e - 6f + g = 0$
7. $a - nb + n \times \frac{n-1}{2} c - n \times \frac{n-1}{2} \times \frac{n-2}{3} d$ &c. $= 0$.

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* The more terms are given the more accurately the equation will approximate.

EXAMPLES.

1. Given the logarithms of 101, 102, 104, and 105; to find the logarithm of 103.

Here the number of terms are 4,

Therefore against 4, *in the table, we have* $a-4b+6c-4d+e=0$; *or* $c=\frac{4\times(b+d)-(a+e)}{6}$.

$$\text{Whence}\begin{cases}a=2.0043214\\ b=2.0086002\\ d=2.0170333\\ e=2.0211893\end{cases}$$

$$4\times(b+d)=16.1025340$$
$$a+e=4.0255107$$

6)12.0770233

2.0128372 = *log. of* 103, *as required.*

2. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50. *Ans.* 3.684033.

3. Given the logarithms of 50, 51, 52, 54, 55, and 56, to find the logarithm of 53.

Anſ. 1.7242758695.

PROMISCUOUS EXAMPLES RELATING TO SERIES.

1. To find the ſum (S) of n terms of the ſeries 1, 2, 3, 4, 5, 6, &c.

Firſt, $1+2+3+4+5$, *&c.* . . . . $n=S$.

And $n+(n-1)+(n-2)+(n-3)+(n-4)$ *&c.* . . . . $1=S$.

Therefore $(n+1)+(n+1)+(n+1)+(n+1)$, *&c.* .... $(n+1)=2S$.

And conſequently $(n+1)\times n=2S$; *or* $S=\frac{n^2+n}{2}=$ *ſum required.*

2. To find the ſum (S) of n terms of the ſeries 1, 3, 5, 7, 9, 11, &c.

Firſt, $1+3+5+7+9$, *&c.* ..... $2n-1=S$.
And $(2n-1)+(2n-3)+(2n-5)+(2n-7)+(2n-9)$, *&c.* ..... $1=S$.
Therefore $2n+2n+2n+2n+2n$, *&c.* ..... $2n=2S$.

And, conſequently, $2n\times n=2S$; *or* $S=\frac{2n\times n}{2}=n^2=$ *ſum required.*

3. Required the ſum (S) of n terms of the ſeries $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$, &c.

Firſt, $a+(a+d)+(a+2d)+(a+3d)$, *&c.* ..... $a+(n-1)\times d=S$.
And $a+(nd-d)+a+(nd-2d)+a+(nd-3d)+a+(nd-4d)$, *&c.* ..... $a=S$.
Therefore $2a+(nd-d)+2a+(nd-d)+2a+(nd-d)$, *&c.* $2a+(nd-d)=2S$.
And, conſequently, $(2a+nd-d)\times n=2S$; *or* $S=(2a+nd-d)\times\frac{n}{2}=$ *ſum required.*

OR THUS:

Firſt, $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$, *&c.*

$$=\left\{\begin{matrix}(+1+1+1+1+1,\ \&c.)\ \times a\\(+0+1+2+3+4,\ \&c.)\ \times d\end{matrix}\right\}=S.$$

But n terms of $1+1+1+1+1$, *&c.* $=n$.

And n terms of $0+1+2+3+4$, *&c.* $=\frac{n\times(n-1)}{2}$

And therefore $S=na+\frac{n\times(n-1)\times d}{2}=(2a+nd-d)\times\frac{n}{2}$ *as before.*

4. To find the ſum (S) of n terms of the ſeries 1, x, x^2, x^3, x^4, &c.

Firſt, $1+x+x^2+x^3+x^4$, *&c.* ..... $x^{n-1}=S$.
And $x+x^2+x^3+x^4+x^5$, *&c.* ..... $x^n=Sx$.
Therefore $-1+x^n=Sx-S$;
Or $S=\frac{x^n-1}{x-1}=$ *ſum required.*

And, when x is a proper fraction, the ſum of the ſeries, continued ad infinitum, may be found in the ſame manner.

Thus, $1+x+x^2+x^3+x^4$, *&c.* $=S$.
And $x+x^2+x^3+x^4+x^5$, *&c.* $=Sx$.
Therefore $-1=Sx-S$; *or* $S-Sx=1$.

Whence $S=\frac{1}{1-x}=$ *ſum of an infinite number of terms, as required.*

5. Required the ſum (S) of the circulating decimal .999999, &c. continued *ad infinitum.*

Firſt, .999999, *&c.* $=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}$, *&c.* $=9\times:\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$, *&c.* $=S$.

Or, $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$, *&c.* $=\frac{S}{9}$.
Therefore, $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}$, *&c.* $=\frac{10S}{9}$.
And, $1=\frac{10S}{9}-\frac{S}{9}=\frac{9S}{9}$.
Whence $S=1=$ *ſum required.*

6. Required the ſum (S) of the ſeries $a^2+(a+d)^2+(a+2d)^2+(a+3d)^2+(a+4d)^2$, &c. continued to n terms.

$$\begin{array}{l} \textit{Firſt, } a^2=a^2 \\ (a+\ d)^2=a^2+2\times 1ad+\ 1d^2 \\ (a+2d)^2=a^2+2\times 2ad+\ 4d^2 \\ (a+3d)^2=a^2+2\times 3ad+\ 9d^2 \\ (a+4d)^2=a^2+2\times 4ad+16d^2 \\ \quad \textit{\&c.} \qquad\qquad\quad \textit{\&c.} \end{array}$$

$$\textit{Therefore, } S=\left\{\begin{array}{l} \textit{Sum of } n \textit{ terms of } (1+1+1+1, \textit{\&c.}) \\ \quad \times a^2 \\ +\ \ldots \textit{ ditto of } (0+1+2+3, \textit{\&c.}) \\ \quad \times 2ad \\ +\ \ldots\ldots \textit{ ditto of } (0+1+4+9, \textit{\&c.}) \\ \quad \times d^2 \end{array}\right.$$

$$\textit{But } n \textit{ terms of } 1+1+1+1, \textit{\&c.} =n$$

$$\textit{Ditto of } 0+1+2+3, \textit{\&c.} =\frac{n\times(n-1)}{1\times 2}$$

$$\textit{And ditto of } 0+1+4+9, \textit{\&c.} =\frac{n\times(n-1)\times(2n-1)}{1\times 2\times 3}$$

$$\textit{Whence } S=n\times a^2+n\times(n-1)\times ad+\frac{n\times(n-1)\times(2n-1)}{1\times 2\times 3}\times d^2=na^2+nad\times(n-1)+\frac{nd^2\times(n-1)\times(2n-1)}{1\times 2\times 3}=$$
ſum required.

7. Required the ſum (S) of the ſeries $a^3+(a+d)^3+(a+2d)^3+(a+3d)^3+(a+4d)^3$, &c. continued to n terms.

$$\begin{array}{l} \textit{Firſt, } a^3=a^3 \\ (a+\ d)^3=a^3+3\times 1a^2d+3\times\ 1ad^2+\ \ 1d^3 \\ (a+2d)^3=a^3+3\times 2a^2d+3\times\ 4ad^2+\ \ 8d^3 \\ (a+3d)^3=a^3+3\times 3a^2d+3\times\ 9ad^2+\ 27d^3 \\ (a+4d)^3=a^3+3\times 4a^2d+3\times 16ad^2+\ 64d^3 \\ (a+5d)^3=a^3+3\times 5a^2d+3\times 25ad^2+125d^3 \\ \quad \textit{\&c.} \qquad\qquad\qquad\qquad \textit{\&c.} \end{array}$$

$$\textit{Therefore, } S=\left\{\begin{array}{l}\textit{Sum of n terms of } (1+1+1+1, \textit{\&c.}) \\ \quad \times a^3 \\ + \ldots \textit{ ditto of } (0+1+2+3, \textit{\&c.}) \\ \quad \times 3a^2d \\ + \ldots \textit{ ditto of } (0+1+4+9, \textit{\&c.}) \\ \quad \times 3ad^2 \\ + \ldots \textit{ ditto of } (0+1+8+27, \textit{\&c.}) \\ \quad \times d^3 \end{array}\right.$$

But n terms of $1+1+1+1$, *&c.* $=n$

Ditto . . . of $0+1+2+3$, *&c.* $=\frac{n\times(n-1)}{1\times 2}$

Ditto . . . of $0+1+4+9$, *&c.* $=\frac{n\times(n-1)\times(2n-1)}{1\times 2\times 3}$

Ditto of $0+1+8+27+64+125$, *&c.* $=\frac{n^4-2n^3+n^2}{2\times 2}$

Conſequently the ſum $S=n\times a^3+\frac{n\times(n-1)\times 3a^2d}{1\times 2}+\frac{n\times(n-1)\times(2n-1)\times 3ad^2}{1\times 2\times 3}+\frac{(n^4-2n^3+n^2)\times d^3}{2\times 2}=$ *ſum required.*

8. Required the ſum (S) of n terms of the ſeries $1+3+7+15+31$, &c.

The terms of this ſeries are evidently equal to 1, $(1+2)$, $(1+2+4)$, $(1+2+4+8)$, *&c. or the ſucceſſive ſums of the geometrical progreſſion* 1, 2, 4, 8, 16, *&c.*

Let, therefore, $a=1$, *and* $r=2$, *and we ſhall have* $a+ar+ar^2+ar^3+ar^4$, *&c.* $=1+2+4+8+16$, *&c.*

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* For an account of figurate numbers, with the methods of finding their ſums, &c. I muſt refer the learner to Simpſon's Algebra, p. 215, where he will find this ſubject fully explained.

But the ſums of 1, 2, 3, 4, *&c. terms of this ſeries are*

$$
\begin{array}{ll}
1. & \dfrac{ar-a}{r-1} = (r-1) \times \dfrac{a}{r-1} \\
2. & \dfrac{ar^2-a}{r-1} = (r^2-1) \times \dfrac{a}{r-1} \\
3. & \dfrac{ar^3-a}{r-1} = (r^3-1) \times \dfrac{a}{r-1} \\
4. & \dfrac{ar^4-a}{r-1} = (r^4-1) \times \dfrac{a}{r-1} \\
 & \&c. \qquad\qquad \&c.
\end{array}
$$

Therefore, $S = \dfrac{a}{r-1} \times \left\{ \begin{array}{l} n \text{ terms of } r+r^2+r^3+r^4, \&c. \\ -n \text{ terms of } 1+1+1+1, \&c. \end{array} \right.$

But $1+1+1+1+1+1+1$, *&c.* $= n$

And $r+r^2+r^3+r^4$, *&c.* $= (r^n-1) \times \dfrac{r}{r-1}$.

Whence $S = (r^4-1) \times \dfrac{r}{r-1} - n) \times \dfrac{r}{r-1} =$ *ſum required.*

9. * Required the ſum (S) of n terms of the ſeries $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{3}+\frac{31}{16}$, &c.

The terms of this ſeries are the ſucceſſive ſums of the geometrical progreſſion $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, *&c.*

Let therefore $a = 1$ *and* $r = 2$, *then will* $a+\dfrac{a}{r}+\dfrac{a}{r^2}+\dfrac{a}{r^3}+\dfrac{a}{r^4}+\dfrac{a}{r^5}+\dfrac{a}{r^6}$, *&c.* $= \dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}+\dfrac{1}{32}+\dfrac{1}{64}+\dfrac{1}{128}$, *&c. continued at pleaſure.*

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* Many queſtions of this kind, as well as ſeveral other things relating to ſeries in general, may be found in Dodſon's Mathematical Repoſitory, which is an excellent analytical performance.

But the ſums of 1, 2, 3, 4, *&c. terms of this ſeries are*

$$1.\ \frac{(r-1)\times a}{(r-1)\times 1} = (r-1) \times \frac{a}{r-1}$$

$$2.\ \frac{(r^2-1)\times a}{(r-1)\times r} = (r-\frac{1}{r}) \times \frac{a}{r-1}$$

$$3.\ \frac{(r^3-1)\times a}{(r-1)\times r^2} = (r-\frac{1}{r^2}) \times \frac{a}{r-1}$$

$$4.\ \frac{(r^4-1)\times a}{(r-1)\times r^3} = (r-\frac{1}{r^3}) \times \frac{a}{r-1}$$

$$\text{\&c.} \qquad \text{\&c.}$$

$$\textit{Therefore } S = \frac{a}{r-1} \times \begin{cases} n \textit{ terms of } r+r+r+r, \text{ \&c.} \\ -n \textit{ terms of } \frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}, \text{ \&c.} \end{cases}$$

$$\textit{But } r+r+r+r, \text{ \&c.} = nr,$$

$$\textit{And } \frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}, \text{ \&c.} = \frac{r^n-1}{(r-1)\times r^{n-1}},$$

$$\textit{Whence } S = \frac{a}{r-1} \times nr - \frac{r^n-1}{(r-1)\times r^{n-1}} = \textit{ſum required.}$$

10. To find the ſum (S) of the infinite ſeries of the reciprocals of the triangular numbers, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c.

$$\textit{Let } \frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}, \text{ \&c. } \textit{ad infinitum} = S.$$

$$\textit{Or, } \frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{2.3}+\frac{1}{2.5}, \text{ \&c.} \ldots\ldots\ldots = S.$$

$$\textit{Then } \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}, \text{ \&c.} \ldots\ldots\ldots = \frac{S}{2}.$$

$$\textit{That is, } (\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5}) \text{\&c.} = \frac{S}{2}.$$

$$\text{Or, }\left\{\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},\ \text{\&c.}\\ -\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5},\ \text{\&c.}\end{array}\right\}=\frac{S}{2}.$$

$$\text{Whence, }\frac{S}{2}=\frac{1}{1};\ \text{or } S=2=\text{ ſum required.}$$

11. * To find the ſum of n terms of the ſeries $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c.

$$\text{Let } z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},\ \text{\&c. to } \frac{1}{n}.$$

$$\text{Then } z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},\ \text{\&c. to } \frac{1}{n}.$$

$$\text{And } z-\frac{1}{1}+\frac{1}{n+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},\ \text{\&c. to } \frac{1}{n+1},$$

---

* The figurate numbers, of which the terms of this and ſeveral other ſeriesare the reciprocals, may be exhibited thus:

Figurate numbers of the		are						
	1ſt. order		1,	1,	1,	1,	1,	&c.
	2d. order		1,	2,	3,	4,	5,	&c.
	3d. order		1,	3,	6,	10,	15,	&c.
	4th. order		1,	4,	10,	20,	35,	&c.
	5th. order		1,	5,	15,	35,	70,	&c.

It may here alſo be remarked that different ſeries are diſtinguiſhed by particular names, according to the nature of their terms.

Thus, a ſeries, whoſe terms continually decreaſe, is called a converging ſeries.

A ſeries, whoſe terms continually increaſe, is called a diverging ſeries.

And a ſeries, whoſe terms neither increaſe nor diminiſh, is called a neutral ſeries.

Thus: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$, &c. converging; $1-2+3-4+5$, &c. diverging; and $1-1+1-1+$, &c. neutral.

$$\text{Therefore } \frac{1}{1}-\frac{1}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}, \text{ \&c. to } \frac{1}{n}-\frac{1}{n+1}$$

$$\text{Or, } \frac{n}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}, \text{ \&c. to } \frac{1}{n.(n+1)}.$$

$$\text{Whence, } \frac{2n}{n+1}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}, \text{ \&c. to } \frac{2}{n.(n+1}.$$

$$\text{Or, } \frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}, \text{ \&c. to } \frac{2}{n.n+1} \text{ (or } n \text{ terms)}$$

$$=\frac{2n}{n+1}= \text{ ſum of the ſeries, or anſwer required.}$$

12. Required the ſum of the infinite ſeries $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, &c.

$$\text{Let } z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \text{ \&c. ad infinitum.}$$

$$\text{Then } z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \text{ \&c. by tranſpoſition.}$$

$$\text{And } 1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}, \text{ \&c. by ſubtraction.}$$

$$\text{Or, } 1-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}, \text{ \&c. by tranſpoſition.}$$

$$\text{And } \frac{1}{2}=\frac{4}{1.4.3}+\frac{6}{2.9.4}+\frac{8}{3.16.5}, \text{ \&c. by ſubtraction.}$$

$$\text{Or, } \frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}, \text{ \&c.}$$

$$\text{Whence } \frac{1}{2}\div 2=\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{4.5.6}, \text{ ad infinitum.}$$

But $\frac{1}{2}\div 2=\frac{1}{4}$; *therefore* $\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$ *continued to infinity, is equal to* $\frac{1}{4}$, *which is the ſum required.*

13. To find the ſum of *n* terms of the ſeries $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6}$, &c.

$$\text{Let } z = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} +, \text{\&c. to } \frac{1}{n.(n+1)}.$$

$$\text{Then } z - \frac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}, \text{\&c. to } \frac{1}{n.(n+1)}.$$

$$\text{And } z - \frac{1}{2} + \frac{1}{(n+1).(n+2)} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8}, \text{\&c. continued to } \frac{1}{(n+1).(n+2)} \text{ terms.}$$

Therefore $\frac{1}{2} - \frac{1}{(n+1).(n+2)} = \frac{2}{1.2.3} + \frac{2}{2.3.4} + \frac{2}{3.4.5}$, (*continued to n terms*) *by ſubtraction.*

Whence $\frac{1}{4} - \frac{1}{2.(n+1).(n+2)} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$, *&c.* (*continued to n terms*) *by diviſion.*

And conſequently, $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$, *continued to n terms is* $= \frac{1}{4} - \frac{1}{2.(n+1).(n+2)} =$ *ſum required.*

14. Required the ſum (S) of the ſeries $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}$, &c. continued *ad infinitum*.

$$\text{Let } x = \frac{1}{2} \text{ and } S = \frac{z}{1+x}.$$

$$\text{Then } \frac{z}{1+x} = x - x^2 + x^3 - x^4 + x^5, \text{\&c.}$$

$$\text{And } z = (1+x) \times (x - x^2 + x^3 - x^4, \text{\&c.})$$

$$\text{Whence, by multiplication}\left\{\begin{array}{l} x-x^2+x^3-x^4+x^5, \text{ \&c.}\\ 1+x\end{array}\right.$$

$$\begin{array}{l} x-x^2+x^3-x^4+x^5, \text{ \&c.}\\ \quad +x^2-x^3+x^4-x^5, \text{ \&c.}\\ \hline \end{array}$$

Whoſe ſum is $=x+0+0+0+0$, *&c.*

Therefore $z=x$.

And $x-x^2+x^3-x^4+x^5$, *&c.* $=\frac{x}{1+x}$.

Or $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}$, *&c.* $=\frac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}=$ *ſum required.*

15. Required the ſum of the infinite ſeries $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}$, &c.

Let $x=\frac{1}{2}$, *and* $S=\frac{z}{(1-x)^2}$.

Then $\frac{z}{(1-x)^2}=x+2x^2+3x^3+4x^4+5x^5$, *&c.*

And $z=(1-x)^2\times(x+2x^2+3x^3+4x^4$, *&c.*

$$\text{Whence, by multiplication}\left\{\begin{array}{l} x+2x^2+3x^3+4x^4, \text{ \&c.}\\ 1-2x+x^2\end{array}\right.$$

$$\begin{array}{l} x+2x^2+3x^3+4x^4, \text{ \&c.}\\ \quad -2x^2-4x^3-6x^3, \text{ \&c.}\\ \qquad\quad +x^3+2x^4, \text{ \&c.}\\ \hline \end{array}$$

Whoſe ſum is $=x+0+0+0$, *&c.*

Therefor $x=z$.

And $x+2x^2+3x^3+4x^4+5x^5$, *&c.* $=\frac{x}{(1-x)^2}$.

Or $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}$, &c. $=\frac{\frac{1}{2}}{(1-\frac{1}{2})^2}=2=$ *ſum required.*

16. Required the ſum (S) of the infinite ſeries $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$, &c.

Let $x=\frac{1}{3}$, *and* $\frac{z}{(1-x)^3}=S$.

Then $\frac{z}{(1-x)^3}=x+4x^2+9x^3+16x^4+25x^5$, *&c.*

And $z=(1-x)^3\times(x+4x^2+9x^3+16x^4$, *&c.*)
$=x+x^2$, *as will be found by actual multiplication.*

Therefore $x+x^2=z$.

And conſequently $x+4x^2+9x^3+16x^4$, *&c.* $=\frac{x\times(1+x)}{(1-x)^3}$.

Or $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$, *&c.* $=\frac{\frac{1}{3}\times(+\frac{1}{3})}{(1-\frac{1}{3})^3}=\frac{3}{2}=1\frac{1}{2}=$ *ſum required.*

17. Required the ſum (S) of the infinite ſeries $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}$, &c.

Let $x=\frac{1}{r}$, *and* $S=\frac{z}{m.(1-x)^2}$.

Then $\frac{z}{m(1-x)^2}=\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{r^3}$, *&c.*

Or $\frac{z}{(1-x)^2}=a+\frac{a+d}{r}+\frac{a+2d}{r^2}+\frac{a+3d}{r^3}$, *&c.*

That is, $\frac{z}{(1-x)^2}=a+(a+d)\times x+(a+2d)\times x^2+(a+3d)\times x^3$, *&c.*

And $z=(1-z)^2\times(a+(a+a)x+(a+2d)x^2+(a+3d)x^3$, *&c.*) $=(1-x)\times a+dx$, *as will appear by actual multiplication.*

Therefore $z=(1-x)\times a+dx$.

And conſequently $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}$, *&c.* $=$ $\frac{(1-x)\times a+dx}{m.(1-x)^2}=$ *ſum of the infinite ſeries required.*

EXAMPLES FOR PRACTICE.

1. To find the ſum of n terms of the ſeries $a+(a-d)+(a-2d)+(a-3d)+(a-4d)$, &c.

Anſ. $\frac{n}{2}\times(2a-(n-1)\times d.)$

2. Required the ſum of the infinite ſeries $a+da+d^2a+d^3a+d^4a$, &c. where d is a proper fraction.

Anſ. $\frac{a}{1-d}$.

3. Required the ſum of the infinite ſeries $1+3x+6x^2+10x^3+15x^4$, &c.

Anſ. $\frac{1}{(1-x)^3}$.

4. Required the ſum of the infinite ſeries $1+4x+10x^2+20x^3+35x^4$, &c.

Anſ. $\frac{1}{(1-x)^4}$.

5. Required the ſum of the infinite ſeries $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}$, &c.

Anſ. $\frac{1}{2}$.

6. Required the ſum of 40 terms of the ſeries $(1\times2)+(3\times4)+(5\times6)+(6\times7)+(7\times8)$, &c.

Anſ. 22960.

7. To find the ſum of the infinite ſeries $1+2^4x+3^4x^2+4^4x^3+5^4x^4$, &c.

Anſ. $\frac{1+11x+11x^2+x^3}{(1-x)^5}$.

8. Required the ſum of the infinite ſeries $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c.

Anſ. $\frac{1}{18}$.

9. Required the ſum of n terms of the ſeries $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c.

Anſ. $\frac{1}{18}-\frac{1}{3.(n+1).(n+2).(n+3)}$.

10. Required the ſum of n terms of the ſeries $\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}$, &c.

Anſ. $\frac{(1-p)\times r}{(r-1)^2}-\frac{np}{r-1}$, *where* $p=\frac{1}{r}$.

11. Required the ſum of the ſeries $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}$, &c. $\frac{1}{n\times(3+n)}$.

Anſ. $\Sigma=\frac{11}{18}$, $S=\frac{n}{3+3n}+\frac{n}{12+6n}-\frac{n}{27+9n}$.

12. Required the ſum of the ſeries $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}$, &c. $\frac{1}{2n(4+2n)}$. *Anſ.* $\Sigma=\frac{3}{16}$, $S=\frac{5n+3n^2}{32+48n+16n^2}$.

13. Required the ſum of the ſeries $\frac{1}{4.8}-\frac{1}{6.10}+\frac{1}{8.12}-$, &c. $\frac{1}{(2+2n).(6+2n)}$.

Anſ. $\Sigma=\frac{1}{48}$, $S=\frac{n}{16+16n}-\frac{n}{36+24n}$.

14. Required the ſum of the ſeries $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}$, &c. $\frac{1}{3n.(4+4n)}$ *Anſ.* $\Sigma=\frac{1}{12}$, $S=\frac{n}{12+12n}$.

15. * Required the ſum of the infinite ſeries $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}$, &c.

Anſ. .69314718, *&c. cr hyp. log. of* 2.

16. Required the ſum of the infinite ſeries $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}$, &c.

Anſ. .78539, *&c. or* $\frac{1}{4}$ *cir. of the circle whoſe diam. is* 1.

17. Required the ſum of the diverging ſeries $\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\frac{5}{6}$, &c.

Anſ. .193147, *&c. or* $\frac{1}{2}+$ *hyp. log. of* 2.

18. Required the ſum of the diverging ſeries $\frac{2^2}{1}-\frac{3^2}{2}+\frac{4^2}{3}-\frac{5^2}{4}$, &c.

Anſ. 1.943147, *&c. or* $\frac{5}{4}+$ *hyp. log. of* 2.

19. Required the ſum of the hyper-geometrical ſeries $1-1+2-6+24-120$, &c. or $1-1\text{A}+2\text{B}-3\text{C}+4\text{D}-5\text{E}$, &c. *Anſ. near approx. value* =.298174.

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* A great variety of ſeries, of different forms, may be found in other authors; but thoſe which are here given will be ſufficient for the learner's practice.

The names of the principal authors, who have written upon this ſubject, are as follow:

Archimedes; Arabes; D'Alembert; Barrow; Briggs; Nicholas, Daniel, John and James Bernoulli; Fermat; De Cartes; Clairaut; Condercet; Cotes; Dodſon; Euler; Emerſon; Fagnanus; Le Grange; Goldbach; Gregory; Halley; Harriot; Huddens; Huygens; Hutton; Kepler; Keil; Landen; Mac Laurin; De Lagney; Leibnitz; Lorgna; Lucas de Burgo; Manfredi; Monmort; De Moivre; Montono; Nichole; Newton; Oughtred; Riccati; Regnald; Saunderſon; Sterling; Sluſius; Simpſon; Brook Taylor; Varignon; Vieta; Wallis; Waring; &c.

For an account of theſe ſeries, with a new method of finding their approximate values, ſee Dr. Hutton's Mathematical Tracts, lately publiſhed.

OF LOGARITHMS*.

Logarithms are numbers ſo contrived and adapted to other numbers, that the ſums and differences of the former ſhall correſpond to, and ſhew, the products and quotients of the latter.

Or, more generally, logarithms are the numerical exponents of ratios, or a ſeries of numbers in arithmetical progreſſion, anſwering to another ſeries of numbers in geometrical progreſſion.

Thus	0,	1,	2,	3,	4,	5,	*Indices, or logarithms.*
	1,	2,	4,	8,	16,	32,	*Geometric progreſſion.*
Or	0,	1,	2,	3,	4,	5,	*Indices, or logarithms.*
	1,	3,	9,	27,	81,	243,	*Geometric progreſſion.*
Or	0,	1,	2,	3,	4,	5,	*Ind. or log.*
	1,	10,	100,	1000,	10000,	100000,	*Geo. prog.*

&c.

And, from hence, it is evident, that there may be as many kinds of indices, or logarithms, as there can be taken different kinds of geometric ſeries; and that in any ſyſtem, or table, of logarithms whatever, the logarithm of unity, or 1, will be nothing.

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* The invention of logarithms is the undoubted right of Lord *Neper*, Baron of *Merchiſton*, in *Scotland*, and is properly conſidered as one of the moſt uſeful and excellent diſcoveries of modern times. A table of theſe numbers was firſt publiſhed by him at Edinburgh, anno 1614, in a treatiſe entitled *Canon Mirificum Logarithmorum*; and, as their great utility and extenſive application were ſufficiently apparent, they were immediately received by all the learned throughout Europe. Mr. *Henry Briggs*, *Savilian* Profeſſor of Geometry at *Oxford*,

It is alſo apparent, from the nature of theſe ſeries, that if any two indices be added together their ſum will be the index of that number which is equal to the product of the two terms, in the geometric progreſſion, to which thoſe indices belong. original

Thus, the indices 2 *and* 3, *being added together, are* =5; *and the numbers* 4 *and* 8, *or the terms correſponding with thoſe indices, being multiplied together, are* =32, *which is the number anſwering to the index* 5.

And, in like manner, if any one index be ſubtracted from another, the difference will be the index of that number, which is equal to the quotient of the two terms to which thoſe indices belong.

Thus the index 6, *minus the index* 4, *is* =2; *and the terms correſponding to thoſe indices are* 64 *and* 16, *whoſe quotient is* =4; *which is the number anſwering to the index* 2.

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upon hearing of the diſcovery, ſet out upon a viſit to the noble inventor, and ſoon afterwards they jointly undertook the arduous taſk of computing new tables upon this ſubject, and reducing them to a more convenient form than that which was at firſt thought of. But Lord *Neper* dying before they were finiſhed, the whole burden was laid upon Mr. *Briggs*, who with prodigious labour, and great ſkill, made an entire *Canon*, according to the new form, for all numbers from 1 to 20000, and from 9000 to 101000, to 14 places of figures, and publiſhed it at *London* in the year 1624, in a treatiſe entitled *Arithmetica Logarithmica*, with directions for ſupplying the intermediate *chiliads*.

This *Canon* was again publiſhed in *Holland*, by *Adrian Vlacq*, *anno* 1628, together with the logarithms of all the numbers which Mr. *Briggs* had omitted; but he continued them only to 10 places of decimals. Mr. *Briggs* alſo computed the logarithms of the ſigns, tangents and ſecants, to every degree, and $\frac{1}{100}$ part of a degree of the whole quadrant; and ſubjoined

For the ſame reaſon, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power. to which ye quantity is to be raised

Thus, the index or logarithm of 4, *in the above ſeries, is* 2; *and if this number be multiplied by* 3, *the product will be* =6; *which is the logarithm of* 64, *or the third power of* 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root.

Thus the index or logarithm of 64 *is* 6; *and if this number be divided by* 2, *the quotient will be* =3; *which is the logarithm of* 8, *or the ſquare root of* 64.

The logarithms moſt convenient for practice are ſuch as are adopted to a geometric ſeries increaſing

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them to the natural ſigns, tangents and ſecants, which he had before computed to 15 places of figures. And theſe tables, together with their conſtruction and uſe, were firſt publiſhed in the year 1633, after Mr. *Briggs*'s death, by Mr. *Henry Gellibrand*, under the title of *Trigonometria Britannica*.

Benjamin Urſinus has alſo given us a table of logarithms to every 10 ſeconds. And Mr. *Welf*, in his *Mathematical Lexicon*, ſays that one *Van Loſer* had computed them to every ſingle ſecond, but his untimely death prevented their publication.

A great number of other authors have treated of this ſubject, but as their numbers are frequently inaccurate, and incommodiouſly diſpoſed, they are now generally neglected. The tables in moſt repute at preſent are thoſe of *Gardiner* in 4to, firſt printed in the year 1742, and *Sherwin* in 8vo, firſt printed in the year 1705, where the logarithms of all numbers may be eaſily found from 1 to 10000000; and thoſe of the ſigns, tangents, and ſecants, to any degree of accuracy required.

Dodſon's Antilogarithmic Canon is likewiſe a very ingenious work, and is of great uſe for finding the numbers anſwering to any given logarithms.

in a tenfold proportion, as in the laſt of the above examples; and are thoſe which are to be found, at preſent, in moſt of the common tables upon this ſubject.

The diſtinguiſhing mark of this ſyſtem of logarithms is, that the index, or logarithm, of 1 is 0; that of 10, 1; that of 100, 2; that of 1000, 3, &c. And in decimals the logarithm of .1 is —1; that of .01, —2; that of .001, —3; &c.

And, from hence, it follows that the logarithm of any number between 1 and 10 muſt be 0 and ſome fractional parts; and that of a number between 10 and 100 will be 1 and ſome fractional parts; and ſo on for any other number whatever.

And ſince the integral part of a logarithm is always thus readily found, it is uſually called the *index*, or *characteriſtic*; and is commonly omitted in the tables; being left to be ſupplied by the operator himſelf, as occaſion requires.

OF THE MAKING

OF LOGARITHMS.

Whatever arithmetical progreſſion we apply to a geometrical one, the terms of it are logarithms only to that ſeries to which we apply them, and anſwer the end propoſed only for thoſe particular numbers; ſo that if we had logarithms adapted only to particular geometrical ſeries, they would be but of little uſe. The great end and deſign of logarithms is the eaſe and expedition which they afford in long calculations, by ſaving the laborious work of *multiplication*, *diviſion*, and the *extraction of roots*; but this end would never

Illustration. ,1; ,01; ,001; = $\frac{1}{10}$; $\frac{1}{10^2}$; $\frac{1}{10^3}$; = 10^{-1}; 10^{-2}; &c

be completely anſwered, unleſs logarithms could be adapted to the whole ſyſtem of numbers, 1, 2, 3, 4, &c. And as here lie the chief excellence and merit of the contrivance, ſo alſo the difficulty. For the natural ſyſtem of numbers, 1, 2, 3, 4, &c. being an arithmetical, and not a geometrical ſeries, ſeems rather fit to be made logarithms of, than to have logarithms applied to it. But this difficulty may be eaſily removed, by conſidering,

That though the whole ſyſtem of natural numbers, 1, 2, 3, 4, &c. is not in geometrical progreſſion, and cannot, by any means, be made to agree with ſuch a ſeries, yet they may be brought ſo near it, as to be within any aſſignable degree of approximation; which may be conceived, in general, thus; ſuppoſe a fraction indefinitely ſmall to be repreſented * by x, and a geometrical ſeries ariſing from 1, in the ratio of 1 to $1+x$, to be 1, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, &c. Then muſt ſome of theſe terms come indefinitely near to all the natural numbers, 1, 2, 3, 4, &c.; becauſe, amongſt numbers which ariſe by extremely ſmall increments, ſome of them muſt exceed, or fall ſhort, of any determinate number, by an indefinitely little exceſs or defect.

If, therefore, in the places of the terms of this ſeries, which approach indefinitely near to any of the natural numbers, we ſuppoſe theſe natural numbers themſelves to be ſubſtituted, then will this ſeries be in geometrical progreſſion, to an exactneſs which may be called *indefinite*; becauſe the approximation of its terms to the natural numbers can never end, but goes on *in infinitum*.

And ſince this imagined geometric ſeries comprehends, indefinitely near, the whole ſyſtem of natural numbers, 1, 2, 3, 4, &c. ſo the indices of its terms comprehend a whole ſyſtem of logarithms, which are

For the less is the value of x, the greater will be it's approx^n
to the natural numbers, 1, 2, 3, &c; suppose $x=9$; $(1+x)=10$
Then by logarithmic involution = 1; 10; 100; 1000; 10000;
$(1+x)=9$; By invol^n = 1; 9; 81; 729; $(1+x)=8$; Invol^n = 1; 8; 64;
$(1+x)=7$; By invol^n = 1; 7; 49; 243; $(1+x)=6$; Invol^n = 1; 6; 36;
$(1+x)=2$; By invol^n = 1; 2; 4; 8; 16; if $x=1$, the great^r will [illegible] approximat^n

adapted to this ſyſtem of numbers, and may be extended to any length we pleaſe. For though the natural ſyſtem of numbers make not, by themſelves, a complete geometrical ſeries, yet they are conceived as a part of ſuch a ſeries, and their logarithms are the indices of their diſtances from unity in that ſeries; or, more generally, they are the correſponding terms of an arithmetical ſeries applied to that geometrical one.

But, again, it muſt be obſerved, that an indefinitely ſmall fraction cannot be aſſigned; and, therefore, in the actual conſtruction of logarithms, we muſt be contented with a determinate degree of approximation. Whence, according as we take x, in the ſeries 1, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, &c. the approximation of its terms to the natural numbers will be in different degrees of exactneſs: for the leſs x is, the nearer will be the approximation; but then the more are the number of involutions of $1+x$, neceſſary to come within any determinate degree of nearneſs to the natural number aſſigned.

Thus then we may conceive the poſſibility of making logarithms to all the natural numbers, 1, 2, 3, 4, &c. to any determinate degree of exactneſs; viz. by aſſigning a very ſmall fraction for x, and actually raiſing a ſeries, in the ratio of 1 to $1+x$, and taking for the natural numbers ſuch terms of that ſeries as are the neareſt to them, and their indices for the logarithms. But then, to conſtruct logarithms in this manner, to ſuch an extent of numbers, and degree of exactneſs, as would be neceſſary to make them of any conſiderable uſe, is next to impoſſible, becauſe of the almoſt infinite labour and time it would require. This, however, is an introduction for underſtanding the method of the *noble inventor*, who undoubtedly firſt took the hint of making logarithms from the conſideration of the indices of a geometrical ſeries; and by means of the principles and known properties of theſe progreſſions he firſt formed his tables, and adapted them to the practical purpoſes intended.

PROBLEM I.

To find the logarithm of any of the natural numbers, 1, 2, 3, 4, *&c. according to the method of* NEPER.

RULE*.

1. Take the geometrical ſeries, 1, 10, 100, 1000, 10000, &c. and apply to it the arithmetical ſeries 1, 2, 3, 4, &c. as logarithms.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the ſeries betwixt which the number propoſed lies.

3. Between the mean, thus found, and the neareſt extreme, find another geometrical mean, in the ſame manner; and ſo on, till you are arrived within the propoſed limit of the number whoſe logarithm is ſought.

4. Find as many arithmetical means, in the ſame order as you ſound the geometrical ones, and the laſt of theſe will be the logarithm anſwering to the number required.

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* The reader who wiſhes to inform himſelf more particularly concerning the hiſtory, nature, and conſtruction of logarithms, may conſult Dr. Hutton's Mathematical Tables, lately publiſhed, where he will find his curioſity amply gratified.

EXAMPLE.

Let it be required to find the logarithm of 9.

Here the numbers between which 9 *lies are* 1 *and* 10.

First, then, the log. of 10 *is* 1, *and the log. of* 1 *is* 0; *therefore* $\frac{1+0}{2}=.5$ *is the arithmetical mean, and* $\sqrt{(1\times10)}=3.1622777=$ *geometric mean: whence the logarithm of* 3 1622777 *is* .5.

Secondly, the log. of 10 *is* 1, *and the log. of* 3.1622777 *is* .5; *therefore* $\frac{1+.5}{2}=.75=$ *arithmetical mean, and* $\sqrt{(10\times3.1622777)}=5.6234132=$ *geometric mean: whence the log. of* 5.6234132 *is* .75.

Thirdly, the log. of 10 *is* 1, *and the log. of* 5.6234132 *is* .75; *therefore* $\frac{1+.75}{2}=.875=$ *arithmetical mean, and* $\sqrt{(10.\times5.6234132)}=7.4989421$ $=$ *geometric mean: whence the log. of* 7.4989421 *is* .875.

Fourthly, the log. of 10 *is* 1, *and the log. of* 7.4989421 *is* .875; *therefore* $\frac{1+.875}{2}=.9375=$ *arithmetical mean, and* $\sqrt{(10\times7.4989421)}=8.6596431$ $=$ *geometric mean: whence the log. of* 8.6596431 *is* .9375.

Fifthly, the log. of 10 *is* 1, *and the log. of* 8.6506431 *is* .9375; *therefore* $\frac{1+.9375}{2}=.96875$ $=$ *arithmetical mean, and* $\sqrt{(10\times8.6596431)}=$ 9.3057204 $=$ *geometric mean: whence the log. of* 9.3057204 *is* 96875.

Sixthly, the log. of 8.6596431 *is* .9375, *and the log. of* 9.3057204 *is* .96875; *therefore* $\frac{.9375+.96875}{2}=$ $.953125=$ *arith. mean, and* $\surd(8.6596431\times9.3057204)$ $=8.9768713=$ *geometric mean: whence the log. of* 8.9768713 *is* .953125.

And, proceeding in this manner, after 25 *extractions, the logarithm of* 8.9999998 *will be found to be* .9542425; *which may be taken for the logarithm of* 9, *becauſe it differs from it only by* $\frac{1}{5000000}$, *and is therefore ſufficiently exact for all practical purpoſes.*

And in the ſame manner the logarithms of almoſt all the prime numbers were found; a work ſo incredibly laborious, that the unremitted induſtry of ſeveral years was ſcarcely ſufficient for its accompliſhment.*

* prime numb. are those wch have no factors

PROBLEM II.

To determine the hyperbolic logarithm (L) *of any given number* (N).

The hyperbolic logarithm of any number is the index of that term of the logarithmic progreſſion, which agrees with the propoſed number, multiplied by the exceſs of the common ratio above unity.

Let, therefore, $(1+x)^n$ be that term of the logarithmical progreſſion, $1, (1+x)^1, (1+x)^2, (1+x)^3, (1+x)^4$, &c. which is equal to the required number (N).

Then will $(1+x)^n=\text{N}$, and $1+x=\text{N}^{\frac{1}{n}}$; and if $1+y$ be put $=\text{N}$, and $m=\frac{1}{n}$, we ſhall have $1+x$ $=\text{N}^{\frac{1}{n}}=(1+y)^m=1+my+m\times\frac{m-1}{2}y^2+m\times\frac{m-1}{2}\times\frac{m-2}{3}y^3$, &c.

It differs from 9 the number sought, by the decimal fraction $\frac{2}{10000000}$ being ÷ by 2 will $=\frac{1}{5000000}$

And, conſequently, $x = my + m \times \frac{m-1}{2} y^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} y^3$, &c. where m being rejected in the factors $m-1$, $m-2$, $m-3$, &c. as being indefinitely ſmall in compariſon of 1, 2, 3, &c. the equation will become $x = my - \frac{my^2}{2} + \frac{my^3}{3} - \frac{my^4}{4}$, &c.

X Hence $\frac{x}{m}$ $(nx = \text{L}) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}$, &c. = hyperbolic logarithm of N, as was required.

PROBLEM III.

The hyperbolic logarithm (L) *of a number being given, to find the number* (N) *itſelf, which anſwers to it.*

Let $(1+x)^n$ be that term of the logarithmic progreſſion, 1, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, &c. which is equal to the required number N.

Then, becauſe $(1+x)^n$ is univerſally $= 1 + nx + n \times \frac{n-1}{2} x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3$, &c. we ſhall alſo have $1 + nx + n \times \frac{n-1}{2} x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3$, &c. $=$ N.

But ſince n, from the nature of logarithms, is here ſuppoſed indefinitely great, it is evident that the numbers connected to it by the ſign — may all be rejected, as far as any aſſigned number of terms.

For as 1, 2, 3, &c. are indefinitely ſmall in compariſon to n, the rejecting of thoſe numbers can very little affect the values to which they belong.

The leſs x is the greater must be its involution de=noted by ye index n; but m, is $= \frac{1}{n}$: Therefore the greater ye denominator is the leſs must be ye value of ye fraction & also ye number which is = to it.

* Illustration. $\frac{x}{m} = \frac{x}{\frac{1}{n}}$ by Substitn. Therefore = nx or L

If, therefore 1, 2, 3, &c. be thrown out of the factors $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. we ſhall have $1+nx+\frac{n^2x^2}{2}+\frac{n^3x^3}{2.3}+\frac{n^4x^4}{2.3.4}$, &c. $=$ N.

But $nx(=\text{L})$ is the hyperbolic logarithm of $(1+x)^n$, or N, by what has been before ſpecified; and therefore $1+\text{L}+\frac{\text{L}^2}{2}+\frac{\text{L}^3}{2.3}+\frac{\text{L}^4}{2.3.4}$, &c. $=\text{N}=$ number required.

PROBLEM IV.

To determine the hyperbolic logarithm (L) *of any given number* (N), *by an univerſally converging ſeries.*

The ſeries $y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}$, &c. is the moſt eaſy and natural that can be obtained; but, in determining the logarithms of large numbers, it is but of little uſe, ſince, in all ſuch caſes, it diverges inſtead of converging.

Let, therefore, the number whoſe logarithm you would find be denoted by $\frac{1}{1-y}$, and alſo let $(1+x)^n$ be the term of the logarithmic progreſſion agreeing with the propoſed number.

Then $(1+x)^n=\frac{1}{1-y}$; or $1+x=\frac{1}{(1-y)}^{\frac{1}{n}}=$

1X $(1-y)^{-\frac{1}{n}}=(1-y)^m$ (by putting $m=-\frac{1}{n}$) $=\overline{1}-my+m\times\frac{m-1}{2}y^2-m\times\frac{m-1}{2}\times\frac{m-2}{3}y^3$, &c. $=\overline{1}+x$ Then

$x=-my-\frac{my^2}{2}-\frac{my^3}{3}$; &c; &c m being reject.d in ye factors $m-1$, $m-2$ &c. By transpos.n $my+\frac{my^2}{2}+\frac{my^3}{3}=-x$

Divid.g the Terms by m. $-\frac{x}{m}=y+\frac{y^2}{2}+\frac{y^3}{3}$ &c

By substit.g the value of m; $-\frac{x}{-\frac{1}{n}}$ or $nx=y+\frac{y^2}{2}+\frac{y^3}{3}$ &c

Whence $y+\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}$, &c. $=-\frac{x}{m}=nx=$ hyperbolic logarithm of $\frac{1}{1-y}$; which ſeries, it is manifeſt, will conſtantly converge, let the value of $\frac{1}{1-y}$ be ever ſo great; becauſe y will always be leſs than unity.

But it is to be obſerved that this ſeries, except in its ſigns, has exactly the ſame form with that above found for the logarithm of $1+y$, and that, if both of them be added together, the ſeries $2y+\frac{2y^3}{3}+\frac{2y^5}{5}+\frac{2y^7}{7}$, &c. thence ariſing, will be more ſimple than either of them, as one half of the terms will, in that caſe, be entirely deſtroyed.

Since, therefore, the ſum of the logarithms of any two numbers is equal to the logarithm of the product of thoſe numbers, it is manifeſt that $2x+\frac{2x^3}{3}+\frac{2x^5}{5}$, &c. will truly expreſs the logarithm of $\frac{1+x}{1-x}$; which ſeries converges ſtill faſter than $x+\frac{x^2}{2}+\frac{x^3}{3}$, &c. not only becauſe the even powers are here deſtroyed, but becauſe x, in finding the logarithm of any given number (N) will have a leſs value.

And, in order to determine what this value of x muſt be, make $\frac{1+x}{1-x}=\text{N}$, and then x will be found $=\frac{\text{N}-1}{\text{N}+1}$; but if the quantity propoſed $\frac{\text{P}}{\text{Q}}$ be a fraction, inſtead of a whole number, make $\frac{1+x}{1-x}=\frac{\text{P}}{\text{Q}}$.

* $N=\frac{1}{1-y}$; $N(1-y)=1$; $N-Ny=1$; Transpos.n $y=\frac{N-1}{N}$

$+nx=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}$ &c $=$ Log of $1+y$ R N.

$nx=y+\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}$ &c $=$ Log of $\frac{1}{1-y}$ R N; R $=(1-y)^{-1}$

$2y+\cdots+\frac{2y^3}{3}+\cdots+\frac{2y^5}{5}$ &c $=$ Log of $(1+y)\times(1-y)^{-1}=\frac{1+y}{1-y}$

Acc. by add.n of ye Logs. you X ye numbers correspond.g to [illegible]. Therefore $N=\frac{1+y}{1-y}$

and you ſhall have $x = \frac{P-Q}{P+Q}$; and either of theſe values being ſubſtituted in the foregoing ſeries $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$, &c. will give the hyperbolic logarithm of the number required.

Now, by finding *Neper*'s logarithm of any number, according to the foregoing method, *Briggs*'s, or the common logarithm of the ſame number, may be found as follows:

Briggs's logarithm of any number is, to *Neper*'s logarithm of the ſame number. as *Briggs*'s logarithm of 10 is to *Neper*'s logarithm of 10.

But *Briggs*'s logarithm of 10 is 1, and *Neper*'s logarithm of 10 is 2.302585093; and, therefore, if *Briggs*'s, or the common logarithm of any number, be denoted by C.L. and *Neper*'s, or the hyperbolic logarithm of the ſame number, by H.L. we ſhall have $2.302585093 : 1 :: \text{H.L.} : \text{C.L}$; or $\text{H.L} \times \frac{1}{2.302585093} = \text{H.L} \times .4342944819 = \text{C.L.}$ as was required *.

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* There are, beſides theſe, many other ingenious methods, which later writers have diſcovered, for finding the logarithms of numbers in a much eaſier way than the original inventor; but, as they cannot be underſtood without a knowledge of ſome of the higher branches of the mathematics, I have thought proper to omit them, and muſt beg leave to refer the reader to thoſe works which are written expreſsly upon the ſubject.

It would likewiſe much exceed the limits of this compendium, to point out all the peculiar artifices that are made uſe of for conſtructing an entire table of theſe numbers; ſuch as thoſe of *Gardiner*, *Sherwin*, and others, who have treated on this ſubject; but any information of this kind, which the learner may wiſh to obtain, may be found in Hutton's Tables, before mentioned.

Continued $= N - Ny = 1 + y$; $Ny + y = N - 1$; $y(N+1) = N-1$;
$y = \frac{N-1}{N+1}$.

OF THE METHOD OF USING

A TABLE OF LOGARITHMS.

Having explained the method of making a table of the logarithms of numbers greater than unity, the next thing to be done is, to ſhew how the logarithms of fractional quantities may be ſound. And, in order to this, it may be obſerved, that as we have hitherto ſuppoſed a geometric ſeries to increaſe from an unit on the right hand, ſo we may now ſuppoſe it to decreaſe from an unit towards the left; and the indices, in this caſe, being made negative, will ſtill exhibit the logarithms of the terms to which they belong.

Thus, Log.	-3	-2	-1	0	$+1$	$+2$	$+3$,	*&c.*
+ *Num.*	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100	1000,	*&c.*

Where $+1$ *is the logarithm of* 10, *and* -1 *the logarithm of* $\frac{1}{10}$; $+2$ *the logarithm of* 100, *and* -2 *the logarithm of* $\frac{1}{100}$, *&c.*

And from hence it appears that all numbers, conſiſting of the ſame figures, whether they be integral, fractional, or mixed, will have the decimal parts of of their logarithms the ſame.

It will be ſufficient to obſerve here that the logarithms of all the prime numbers being had, thoſe of the compoſite numbers may be found only by means of addition and ſubtraction. Thus, $L.4 = 2L.2$; $L.10 = L.2 + L.5$; $L.5 = L.10 - L.2$; $L.6 = L.2 + L.3$, and ſo on for any other of theſe numbers.

In like manner, the log. of $a \times b = L.a + L.b$; the log. of $a \div b = L.a - L.b$; the log. of $r^n = nL.r$ and the log. of $r^{\frac{1}{n}} = \frac{1}{n}L.r$. i, e; by mak'g $r = 100$; $\frac{1}{n} = \frac{1}{3}$; $\sqrt[3]{10^2}$; $= 10^{\frac{2}{3}}$; ie; $= \frac{1}{3} \times$ Lo 10

* $\frac{1}{1000}$; $\frac{1}{100}$; $\frac{1}{10}$; $= 10^{-3}$; 10^{-2}; 10^{-1}; $= \frac{1}{10^3}$; $\frac{1}{10^2}$; $\frac{1}{10}$; 0

Thus, the logarithm of 5874 *being* 3·7689339, *the logarithms of* $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, *&c. part of it will be as follows:*

Num.	Logarithms.
5874	3·7689339
587·4	2·7689339
58·74	1·7689339
5·874	0·7689339
·5874	—1·7689339
·05874	—2·7689339
·005874	—3·7689339

From this it alſo appears, that the *index*, or *characteriſtic*, of any logarithm, is always one leſs than the number of figures which the natural number conſiſts of; and this index is conſtantly to be placed on the left hand of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a line drawn before it, like a negative quantity in algebra.

Thus, a number having 1, 2, 3, 4, 5, *&c. integer places.*

The index of its log. is 0, 1, 2, 3, 4, *&c.*

And a fraction having a digit in the place of primes, ſeconds, thirds, fourths, &c.

The index of its logarithm will be —1, —2, —3, —4, *&c.*

It may alſo be obſerved, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative; and all operations are performed by them, in the ſame manner as by negative and affirmative quantities in algebra.

In taking out of a table the logarithm of any number, not exceeding 10000, we have the decimal part by inſpection; and if to this the proper characteriſtic be affixed, it will give the complete logarithm required.

But if the number, whoſe logarithm is required, be above 10000, then find the logarithm of the two neareſt numbers to it that can be found in the table, and ſay, as their difference : the difference of their logarithms : : the difference of the neareſt number and that whoſe logarithm is required : the difference of their logarithms, *nearly*; and this difference being added to, or ſubtracted from, the neareſt logarithm, according as it is greater or leſs than the required one, will give the logarithm required, *nearly*.

Thus, let it be required to find the logarithm of 367182 *.

The decimal part of 3671 *is, by the table* 5647844; *and of* 3672 *is* .5649027;

∴ *The log. of* { 367100 *is* 5.5647844 }
{ 367200 *is* 5.5649027 }

Their diff. 100 .0001183 *diff.*

Nearest N° { 367200
Given N° { 367182

18 *diff.*

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* This method, being founded on the ſuppoſition that the logarithms of all numbers between 367100 and 367200, increaſe, or decreaſe, equally, according to their diſtance from 367100 or 367200, is not ſtrictly true, but nearly ſo; and the greater any numbers are with reſpect to their difference, the nearer will thoſe differences be proportional. And, therefore, though this will not give the exact logarithm, yet it will be a very near approximation, and is ſufficiently exact for moſt practical purpoſes.

Therefore 100 : .0001183 : : 18 : .0000212,

And 5.5649027 — .0000212=5.5648815= *logarithm of* 367182 *nearly.*

If the number conſiſts both of integers and fractions, or is entirely fractional, find the decimal part of the logarithm as if all its figures were integral; and this, being prefixed to the proper characteriſtic, will give the logarithm required.

And if the given number is a proper fraction, ſubtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm ſought; which, being that of a decimal fraction, muſt always have a negative index.

And if it is a mixed number, reduce it to an improper fraction, and find the difference of the logarithms of the numerator and denominator, in the ſame manner as before.

In finding the number anſwering to any given logarithm, the index, if *affirmative*, will always ſhew how many integral places the required number conſiſts of; and, if *negative*, in what place of decimals the firſt, or ſignificant figure, ſtands; ſo that, if the logarithm can be found in the table, the number anſwering to it will always be had by inſpection.

But, if the logarithm cannot be exactly found in the table, find the next greater, and the next leſs, and then ſay, As the difference of theſe two logarithms : the difference of the numbers anſwering to them : : the difference of the given logarithm and the neareſt tabular logarithm : a fourth number; which added to, or ſubtracted from, the natural number anſwering to the neareſt tabular logarithm, according as that logarithm is leſs or greater than the given one, will give the number required, *nearly.*

Thus, let it be required to find the natural number anſwering to the logarithm 5.5648815.

The next leſs and greater logarithms, in the table, are

5.5647844	*The numbers*	367100
5.5649027	*anſwering*	367200
Their diff. .0001183		100 *diff.*

And 5.5649027 — 5.5648815 = .0000212.

Therefore .0001183 : 100 : : .0000212 : *nearly.*

Whence 367200 — 18 = 367182 = *number required* *.

MULTIPLICATION BY LOGARITHMS.

RULE.

Add the logarithms of the factors together, and their ſum will be the logarithm of the product required.

Obſerving to add what is to be carried from the decimal part of the logarithm to the ſum of the affirmative indices:

And that the difference between the affirmative and negative indices is to be taken for the index to the logarithm of the product.

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* Directions, at large, for the uſing of logarithms, may be found in moſt of the common tables upon this ſubject.— *Sherwin*'s Mathematical tables, of the *Edition* 1741 or 1742, are reckoned the moſt correct and convenient, for practical purpoſes, of any now extant, except thoſe of Dr. Hutton, lately publiſhed; which, beſides their accuracy, are much better arranged; and in the two firſt degrees the ſigns, &c. are given to every ſecond. He has alſo a table of hyperbolic logarithms, and ſeveral others equally uſeful.

EXAMPLES.

1. Let the number 256 be multiplied by 4.

The log. of 256 = 2.4082400
The log. of 4 = 0.6020600

The product =1024 ...3.0103000

2. Let the number 8.5 be multiplied by 10.

The log. of 8.5 = 0.9294189
The log. of 10 = 1.0000000

The product =85 .... 1.9294189

3. Let the number 46.75 be multiplied by .3275.

The log. of 46.75 = 1.6697816
The log. of .3275 = —1.5152113

The product =15.31 .... 1.1849929

4. Multiply 3.768, 2.053, and .007693 continually together.

The log. of 3.768 = 0.5761109
The log. of 2.053 = 0.3123889
The log. of .007693 = —3.8860957

The product = .059511 ... —2.7745955

5. Multiply .5, .4, and .12, continually together.

The log. of .5 = —1.6989700
The log. of .4 = —1.6020600
The log. of ,12 = —1.0791812

The product = .024 ... —2.3802112

DIVISION BY LOGARITHMS.

RULE.

From the logarithm of the dividend ſubtract the logarithm of the diviſor, and the number agreeing to the remainder will be the quotient required.

But obſerve to change the index of the diviſor from negative to affirmative, or from affirmative to negative, and then the difference of the affirmative indices muſt be taken for the index to the logarithm of the quotient.

And, alſo, when an unit is borrowed, in the left hand place of the decimal part of the logarithm, add it to the index of the diviſor; but if it be negative ſubtract it; and let the index ariſing from thence be changed and worked with as before.

EXAMPLES.

1. Let the number 56 be divided by the number 4.

The log. of 56 = 1.7481880
The log. of 4 = 0.6020600

The quotient = 14 .... 1.1461280

2. Let the number 50.75 be divided by the number .25.

The log. of 50.75 = 1.7054360
The log. of .25 = −1.3979400

The quotient = 203 ...... 2.3074960

3. Let the number 24 be divided by the number 80.

The log. of .24 = —1.3802112
The log. of 80 = 1.9030900

The quotient .003 .... —3.4771212

4. Let the number .01265 be divided by the number .35.

The log. of .01265 = —2.1020905
The log. of .35 = —1.7403627

The quotient =.023 .... —2.3617278

INVOLUTION BY LOGARITHMS*.

RULE.

1. Seek the logarithm of the given number in the table.

2. Multiply the logarithm, thus found, by the index of the propoſed power.

3. Find the number correſponding to the product, and it will be the power required.

Note. In multiplying a logarithm with a negative Index, by any affirmative number, the product will always be negative:

But what is to be carried from the decimal part of the logarithm will always be affirmative:

And therefore their difference will be the index of the product; and is conſtantly to be made of the ſame kind with the greater.

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* The rule of proportion is performed by adding the logarithms of the two laſt terms, and ſubtracting the logarithm of the firſt.

EXAMPLES.

1. Required the ſecond power of the number 3.874.

The log. of 3.874 = 0.5881596
The index = 2

The power = 15.01 .... 1.1763192

2. Required the third power of the number 2.768.

The log. of 2.768 = 0.4421661
The index = 3

The power = 21.21 .... 1.3264983

3. Required the third power of the number .7916.

The log. of .7916 = —1.8985058
The index = 3

The power = .4961...—1.6955174

4. Required the twelfth power of the number 1.539.

The log. of 1.539 = 0.1872386
The index = 12

The power = 176.6 .... 2.2468632

5. Required the 20th power of 1.05.

Anſ. 2.6533, &c.

6. Required the 100th power of 1.05.

Anſ. 131.50, &c.

EVOLUTION BY LOGARITHMS.

RULE.

1. Seek the logarithm of the given number in the table.

2. Divide the logarithm, thus found, by the denominator of the index of the root propoſed.

3. Find the number correſponding to this quotient, and it will be the root required.

Note. When the index of the logarithm, to be divided, is negative, and does not exactly contain the diviſor, increaſe it by ſuch a number as will make it exactly diviſible, and carry the units borrowed, as ſo many tens, to the left-hand place of the decimal, and then divide as in whole numbers.

EXAMPLES.

1. Required the ſquare root of the number 225.

The log. of 225 = 2.3521825
Therefore 2)2.3521825

The root = 15 .... 1.1760912

2. Required the ſquare root of the number 1501.

The log. of 1501 = 3.1763807
Therefore 2)3.1763807

The root = 38.74 .... 1.5881903

3. What is the cube root of the number .166375?

The log. of .166375 = —1.2210881
Therefore 3)—1.2210881

The root = .55 . . . . —1.7403627

4. What is the ſquare root of the number .08162?

The log. of .08162 = —2.9117966
Therefore 2)—2.9117966

The root = .2857 . . . —1.4558983

5. What is the twelfth root of the number 176.6?

The log. of 176.6 = 2.2469907
Therefore 12)2.2469907

The root = 1.539 1872492

MISCELLANEOUS QUESTIONS.

1. A perſon being aſked what o'clock it was, anſwered, that it was between 8 and 9, and that the hour and minute hands were exactly together; what was the time?

Anſ. $8^{h} : 43' : 38'' \frac{2}{11}$.

2. Divide the number 50 into two ſuch parts, that $\frac{3}{4}$ of one part, added to $\frac{5}{6}$ of the other, may make 40.

Anſ. 20 *and* 30.

3. What two numbers are thoſe, whoſe difference is 12, and their ſquares equal to each other?

Anſ. +6 *and* —6.

4. There is a certain number, conſiſting of two places, which is equal to the difference of the ſquares of its digits; and if 36 be added to it the digits will be inverted; quære the number?

Anſ. 48.

5. Given $x^3+y^2=31$, and $y^3+x^2=17$; to find x and y. *Anſ.* $x=3$ *and* $y=2$.

6. Given $y^3-xy=666$, and $x^3+xy=406$; to find x and y. *Anſ.* $x=7$ *and* $y=9$.

7. Given the ſum of three numbers, in harmonical proportion, $=26$, and their continued product $=576$; to find the numbers. *Anſ.* 12, 8, *and* 6.

8. What two numbers are thoſe, whoſe difference, ſum, and product, are to each other as the numbers 2, 3, and 5, reſpectively? *Anſ.* 2 *and* 10.

9. * To find that number whoſe cube being ſubtracted from its ſquare ſhall leave the greateſt remainder poſſible? *Anſ.* $\frac{2}{3}$.

10. It is required to find the leaſt 3 whole numbers, ſo that $\frac{3}{8}$ of the firſt, $\frac{5}{14}$ of the ſecond, and $\frac{7}{20}$ of the third, ſhall be all equal to each other.

Anſ. 280, 294, *and* 300.

11. Given $zx^3+xz^3=290$, and $x^4+z^4=641$; to x and z. *Anſ.* $x=5$, *and* $z=2$.

12. Given the ſum of three numbers in continued geometrical progreſſion $=39$, and the ſum of their ſquares $=819$: to find the numbers. *Anſ.* 3, 9, 27.

13. Required the leaſt number of weights, and the weight of each, that will weigh from one pound to 29 hundred weight.

Anſ. 1, 3, 9, 27, 81, 243, 729, *and* 2187.

14. Required two numbers ſuch, that their ſum ſhall be equal both to their product and the difference of their ſquares. *Anſ.* 2.618034 *and* 1.618034.

15. It is required to find 4 affirmative integers ſuch, that the ſquare of the greateſt may be equal to the ſum of the ſquares of the other three.

Anſ. 3, 4, 12, *and* 13.

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* This is properly a queſtion in fluxions, but it is anſwered algebraically by Mr. Emerſon, as well as ſeveral others of the ſame nature.

16. If money be lent, at three pet cent.
To thoſe who chuſe to borrow,
In what time ſhall I be worth a pound,
If I lend a crown to-morrow?

Anſ. 46.90036 *years, allowing comp. int.*

17. Required the two leaſt nonquadrate numbers, x and y, ſuch, that x^2+y^2 and x^3+y^3 ſhall be both ſquare numbers. *Anſ.* $x=364$, *and* $y=273$.

18. There are three numbers in geometrical proportion ſuch, that, if the mean be ſubtracted from the ſum of the two extremes, the remainder multiplied by the ſum of the ſaid two extremes will be 91; but, if that remainder be multiplied by the ſum of all the three numbers, the product will be 133; it is required to find the three numbers by a ſimple equation.

Anſ. 4, 6, *and* 9.

19. To determine two numbers whoſe ſum ſhall be a cube, but their product and quotients ſquares.

Anſ. 4 *and* 14, 100 *and* 25, 900 *and* 100.

20. Required that arithmetical progreſſion whoſe number of terms is 10, ſum of the terms 185, and the ſum of the cubes of the terms 104525.

Anſ. 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

21. To divide a given number (N) into 4 ſuch parts, that if any other number (n) be added to the firſt part, deducted from the ſecond, multiplied by the third, and the fourth part divided by it, the ſum, difference, product, and quotient, ſhall be all equal to each other.

Anſ. $\frac{Nn}{(n+1)^2}-n$, $\frac{Nn}{(n+1)^2}+n$, $\frac{N}{(n+1)^2}$ *and* $\frac{Nn\times n}{(n+1)^2}$ [$=$ *parts required.*

22. Given $x^3y+y^3x=512500$, and $x^2y-y^2x=2500$; to find x and y. *Anſ.* $x=25$ *and* $y=20$.

23. Given $x+y+z=6$, $xy+xz+yz=11$, and $xyz=6$; to find x, y, and z. *Anſ.* $x=3$, $y=1$, *and* $z=2$.

24. To find two numbers in the ratio of 5 to 7, which, being reſpectively divided by 9 and 13, ſhall leave 3 and 8 for remainders. *Anſ.* 210 *and* 294.

25. To find three numbers ſuch, that $\frac{1}{2}$ the firſt, $\frac{1}{3}$ of the ſecond, and $\frac{1}{4}$ of the third, ſhall be $=62$; $\frac{1}{3}$ of the firſt, $\frac{1}{4}$ of the ſecond, and $\frac{1}{5}$ of the third $=47$; and $\frac{1}{4}$ of the firſt, $\frac{1}{5}$ of the ſecond, and $\frac{1}{6}$ of the third $=38$. *Anſ.* 24, 60, *and* 120.

26. A, B, and C, are to ſhare 100,000 pounds between them, in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, reſpectively; but C's part being loſt by death, it is required to divide the whole ſum properly between the other two. *Anſ.* A's *ſhare is* $57142\frac{282}{329}$ *l. and* B's $42857\frac{47}{329}$ *l.*

27. To find four numbers, x, y, x and w, having the product of every three given; viz. $xyz=231$, $xyw=420$, $yzw=1540$, and $xzw=660$.

Anſ. $x=3$, $y=7$, $z=11$, *and* $w=20$.

28. To find four numbers in geometric proportion, whoſe ſum is 15, and the ſum of their ſquares 85.

Anſ. 1, 2, 4, 8.

29. To find three numbers, x, y, and z, when the product of each by the ſum of the other two are given; viz. $x\times(y+z)=48$, $y\times(x+z)=39$, and $z\times(x+y)=63$. *Anſ.* $x=4$, $y=3$, *and* $z=9$.

30. What number is that, which, being any how divided, the ſquare of one part, when added to the other part, ſhall always be a ſquare number?

Anſ. 1 *only.*

31. Given $y^3+z=127$, $y^3+x=135$, and $x^3+y^3+z^3=1133$; to find x, y, and z.

Anſ. $x=10$, $y=5$, *and* $z=2$.

32. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2+xz=580$; to find x, y, and z.

Anſ. $x=9$, $y=3$, *and* $z=20$.

33. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$; to find x, y, z.

Anſ. $x=10$, $y=17$, *and* $z=22$.

34. To find the leaſt number, which, being divided by 6, 5, 4, 3, and 2, ſhall leave the remainders 5, 4, 3, 2, and 1, reſpectively. *Anſ.* 59.

35. To find three numbers ſuch, that the ſum or difference of any two of them ſhall be ſquare numbers. *Anſ.* 1873432, 2399057, *and* 2288168.

36. To find two ſquare numbers ſuch, that their ſum may be a ſquare, and their difference a cube, and the ſide of the ſaid ſquare and cube equal to each other. *Anſ.* $\frac{783}{15625}$ *and* $\frac{414}{15625}$.

37. To determine the number of fifteens that can be made out of a common pack of 52 cards.

Anſ. 17264.

38. To find a fraction ſuch, that being taken from its reciprocal the remainder ſhall be a ſquare.

Anſ. Find ſuch a fraction as that its biquadrate being added to 4 is a ſquare, and it will anſwer the queſtion.

39. Given $x^2+xy+y^2=1087$, and $x^4+x^3y^3+y^4=4577295$; to find x and y.

Anſ. $x=21$ *and* $y=17$.

40. Given $x+y+z=78$, $x^2+y^2+z^2=2546$, and $xy-xz-yz=527$; to find x, y, and z.

Anſ. $x=41$, $y=28$, *and* $z=9$.

41. Given $x+y=152$, and $(x-y)^{\frac{2}{3}}\times(x-y)^{\frac{3}{2}}=8192$; to find x and y. *Anſ.* $x=108$, *and* $y=44$.

42. To find three numbers ſuch, that, if to the ſquare of each the product of the other 2 be added, the ſums ſhall be ſquares. *Anſ.* 73, 9, 328.

43. Let the number of cards in a pack (p) be diſtributed into any number of heaps (n), by laying as many cards upon the bottom heap as are ſufficient

to make up its number q; then, by having the number of cards remaining in the dealer's hand, (r) and the number of heaps (n) given, it is required to find the ſum of all the bottom cards.

Anſ. $(q+1)\times n+(r-p)=$ *ſum required.*

44. To find 3 numbers ſuch, that if each be ſubtracted from the cube of their ſum the remainders ſhall be cubes. *Anſ.* $\frac{13851}{85184}$, $\frac{19467}{85184}$, *and* $\frac{12954}{85184}$.

45. Given the cycle of the ſun 18, the golden number 8, and the Roman indiction 10; to find the year. *Anſ.* 1717.

46. To find 3 cube numbers ſuch, that their ſum ſhall be both a ſquare and a cube number; and if that ſum be ſquared it ſhall be a cube, and if it be cubed it ſhall be a ſquare.

Anſ. $\frac{1}{8}x^6$, $\frac{8}{27}x^6$, $\frac{125}{216}x^6$; *where* x *may be any number whatever; if it be* $=216$ *they will be whole numbers.*

47. To find 3 numbers ſuch, that if each be added to the cube of their ſum their ſums ſhall be cubes.

Anſ. $\frac{23625}{156464}$, $\frac{1538}{157464}$, $\frac{18577}{157464}$.

48. With guineas and moidores, the feweſt, which way,
Three hundred and fifty-one pounds ean I pay?
If paid every way 'twill admit of, what ſum
Do the pieces amount to? — my fortune's to come.

Anſ. 9 *guineas and* 253 *moidores; and* 37 *ways, which is* $=12987\,l$.

49. Given $x^{z^{\frac{1}{2}}}=z^{x^{\frac{2}{3}}}=100$; to find x and z.

Anſ. $x=47.706$ *and* $z=1.42$.

50. Given $44000x^2+1=z^2$; to find x and z in whole numbers.

Anſ. $x=40482981221781$ *and* $z=8491781142001$.

51. To find three whole numbers ſuch, that the exceſs of the greateſt above the middle number ſhall be to the exceſs of the middle number above the leaſt, as 3 to 1; and alſo that the ſum of every two of theſe ſhall be ſquares.

Anſ. 1362, 402, 82; *or* $4^n\times 1362$, $4^n\times 402$, *and* $4^n\times 82$; *where* n *is any affirmative integer.*

52. Given $x+y=a(2)$, and $x^9+y^9=b(32)$, to find x and y by quadratics.

Anſ. $x=1.4697175$ *and* $y=.5302824$.

53. Given $x^y=500$, and $y^x=300$; to find x and y.

Anſ. $x=4.6914$ *and* $y=5.5102$.

54. Given $xy\times(x+z)^2=300$, $xz\times(y+z)^2=1296$, and $yz\times(x+y)^2=432$; to find x, y, and z.

Anſ. $x=1$, $y=3$, $z=9$.

55. Given $w^3+x+y+z=57$, $w+x^3+y+z=2763$, $w+x+y^3+z=1350$, and $w+x+y+z^3=153$; to find x, y, z, and w.

Anſ. $x=14$, $y=11$, $z=5$, *and* $w=3$.

56. Given $x+y=1750$, $xz+yv=22708$, $xv+yz=12292$, and $xzv+vzy=159252$; to find x, y, z, and v. *Anſ.* $x=1743$, $y=7$, $z=13$, *and* $v=7$.

57. Given $5x+7y+9z=93256$; to find all the different ſolutions in affirmative integers which the equation will admit of. *Anſ.* 13801148.

58. To find a ſquare number ſuch, that the ſum of all its aliquot parts ſhall be a ſquare number.

Anſ. 2401.

59. To find two ſquare numbers ſuch, that either of them, when added to its aliquot parts, ſhall make the ſame ſum. *Anſ.* 106276 *and* 165649.

60. To find three cube numbers ſuch, that their ſum may be both a ſquare and a cube number.

Anſ. 1, $\frac{2048383}{274625}$, *and* $\frac{15252992}{274626}$.

61. To find 4 whole numbers ſuch, that the difference of every two ſhall be a ſquare number.

Anſ. 1873432, 2288168, 2399057, *and* 6560657.

62. To find three numbers ſuch, that if their ſum be multiplied by the firſt it ſhall be a triangular number, by the ſecond a ſquare, and by the third a cube.

Anſ. $\frac{3}{6}$, $\frac{25}{6}$, *and* $\frac{8}{6}$.

63. To find three biquadrate numbers, the ſum of which ſhall be a ſquare. *Anſ.* 12^4, 15^4 *and* 20^4.

64. To find a right-angled triangle ſuch, that its perimeter ſhall be a cube, and the perimeter together with the area a ſquare.

Anſ. Perp. $= \frac{48406}{2475}$, *baſe* $= \frac{44352}{2475}$, *hyp.* $= \frac{65648}{2475}$.

65. To find two equilateral triangles ſuch, that their areas and perimeters ſhall be both equal.

Anſ. Sides of the 1ſt = 29, 29, 40.
Ditto of the 2d = 37, 37, 24.

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THE END.